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MECHANICS AND HYDROSTATICS
FOR BEGINNERS

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MECHANICS
AND
HYDROSTATICS
FOR BEGINNERS

BY
S. L. LONEY

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PREFACE

THIS little book is of a strictly elementary character, and is intended for the use of students whose knowledge of Geometry and Algebra is not presumed to extend beyond the first two Books of Euclid and the solution of simple Quadratic Equations.

A student who is not acquainted with the first few propositions of Euclid's Sixth Book and the elements of Trigonometry, is recommended to commence with the Appendix at the end of the book. In this Appendix will be found the very few propositions in Elementary Trigonometry that are used in the text.

A few articles, with an asterisk prefixed, may be omitted on a first reading, and the Test Examination Papers may be taken at the end of the chapters to which they refer.

For any corrections, or suggestions for improvement, I shall be grateful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE,
EGHAM, SURREY.
December, 1892.

THE Fourth Edition (being the nineteenth impression) has been thoroughly revised and many additions, including a set of Miscellaneous Examples, have been made.

S. L. L.

December, 1923.

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MECHANICS AND HYDROSTATICS

FOR BEGINNERS

STATICS

CHAPTER I

INTRODUCTION

1. THE present book is divided into three portions.

The first portion will treat of the action of forces on bodies, the forces being so arranged that the bodies are at rest. This is the subject of **Statics**.

The second portion will deal with the action of forces on bodies in motion. This is the subject of **Dynamics**.

The third portion will deal with the properties of liquids and gases and of the effect of forces on them when they are at rest. This is the subject of **Hydrostatics**.

The title Dynamics is often used to include all three of these subdivisions.

2. A Body is a portion of matter limited in every direction.

3. Force is anything which changes, or tends to change, the state of rest, or uniform motion, of a body.

4. Rest. A body is said to be at rest when it does not change its position with respect to surrounding objects.

5. A Particle is a portion of matter which is indefinitely small in size, or which, for the purposes of our investigations, is so small that the distances between its different parts may be neglected.

6. A Rigid Body is a body whose parts always preserve an invariable position with respect to one another.

In nature no body is perfectly rigid. Every body yields, perhaps only very slightly, if force be applied to it. If a

rod, made of wood, have one end firmly fixed and the other end be pulled, the wood stretches slightly; if the rod be made of iron the deformation is very much less.

To simplify our enquiry we shall assume, in the first two divisions of our subject, that all the bodies with which we have to deal are perfectly rigid.

7. Equal Forces. Two forces are said to be equal when, if they act on a particle in opposite directions, the particle remains at rest.

8. Mass. The mass of a body is the quantity of matter in the body. The unit of mass used in England is a pound and is defined to be the mass of a certain piece of platinum kept in the Exchequer Office.

Hence the mass of a body is two, three, four... lbs., when it contains two, three, four... times as much matter as the standard piece of platinum.

In France, and other foreign countries, the theoretical unit of mass used is a gramme, which is equal to about 15.432 grains. The practical unit is a kilogramme (1000 grammes), which is equal to about 2.2046 lbs.

9. Weight. The idea of weight is one with which everyone is familiar. We all know that a certain amount of exertion is required to prevent any body from falling to the ground. The earth attracts every body to itself with a force which, as we shall see in Dynamics, is proportional to the mass of the body.

The force with which the earth attracts any body to itself is called the weight of the body.

10. Measurement of Force. We shall choose, as our unit of force in Statics, the weight of one pound. The unit of force is therefore equal to the force which would just support a mass of one pound when hanging freely.

We shall find in Dynamics that the weight of one pound is not quite the same at different points of the earth's surface.

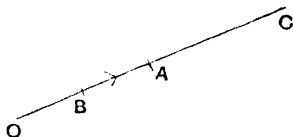
In Statics, however, we shall not have to compare forces at different points of the earth's surface, so that this variation in the weight of a pound is of no practical importance; we shall therefore neglect this variation and assume the weight of a pound to be constant.

11. In practice the expression "weight of one pound" is, in Statics, often shortened into "one pound." The student will therefore understand that "a force of 10 lbs." means "a force equal to the weight of 10 lbs."

12. *Forces represented by straight lines.* A force will be completely known when we know (i) its magnitude, (ii) its direction, and (iii) its point of application, i.e. the point of the body at which the force acts.

Hence we can conveniently represent a force by a straight line drawn through its point of application; for a straight line has both magnitude and direction.

Thus suppose a straight line OA represents a force, equal to 10 lbs. weight, acting at a point O . A force of 5 lbs. weight acting in the same direction would be represented by OB , where B bisects the distance OA , whilst a force, equal to 20 lbs. weight, would be represented by OC , where OA is produced till AC equals OA .



An arrowhead is often used to denote the direction in which a force acts.

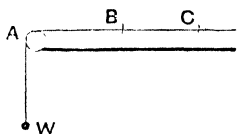
13. *Subdivisions of Force.* There are three different forms under which a force may appear when applied to a mass, viz. as (i) an attraction, (ii) a tension, and (iii) a reaction.

14. *Attraction.* An attraction is a force exerted by one body on another without the intervention of any visible instrument and without the bodies being necessarily in contact. The only example we shall have in this book is the attraction which the earth has for every body; this attraction is (Art. 9) called its weight.

15. *Tension.* If we tie one end of a string to any point of a body and pull at the other end of the string, we exert a force on the body; such a force, exerted by means of a string or rod, is called a **tension**.

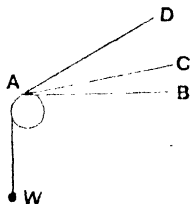
If the string be light [i.e. one whose weight is so small that it may be neglected] the force exerted by the string is the same throughout its length.

For example, if a weight W be supported by means of a light string passing over the edge of a table it is found that the same force must be applied to the string whatever be the point, A , B , or C , of the string at which the force is applied.



Now the force at A required to support the weight is the same in each case; hence it is clear that the effect at A is the same whatever be the point of the string to which the tension is applied, and that the tension of the string is therefore the same throughout its length.

Again, if the weight W be supported by a light string passing round a smooth peg A , it is found that the same force must be exerted at the other end of the string whatever be the direction (AB , AC , or AD) in which the string is pulled, and that this force is equal to the weight W .



[These forces may be measured by attaching the free end of the string to a spring-balance.]

Hence *the tension of a light string passing round a smooth peg is the same throughout its length.*

If two or more strings be knotted together the tensions are not necessarily the same in each string.

The student must carefully notice that the tension of a string is not proportional to its length. It is a common error to suppose that the longer a string the greater is its tension; it is true that we can often apply our force more advantageously if we use a longer piece of string, and hence a beginner often assumes that, other things being equal, the longer string has the greater tension.

16. Reaction. If one body lean, or be pressed, against another body, each body experiences a force at the point of contact; such a force is called a **reaction**.

The force, or action, that one body exerts on a second body is equal and opposite to the force, or reaction, that the second body exerts on the first.

This statement will be found to be included in Newton's Third Law of Motion (Art. 171).

Examples. If a cube of wood is laid on a table, the force

which it exerts on the table is equal and opposite to the force which the latter exerts on it.

If a ladder lean against a wall, the force exerted by the end of the ladder upon the wall is equal and opposite to that exerted by the wall upon the end of the ladder.

17. Equilibrium. When two or more forces act upon a body and are so arranged that the body remains at rest, the forces are said to be in equilibrium.

18. Smooth bodies. If we place a piece of smooth polished wood, having a plane face, upon a table whose top is made as smooth as possible we shall find that, if we attempt to move the block along the surface of the table, some resistance is experienced. There is always some force, however small, between the wood and the surface of the table.

If the bodies were perfectly smooth there would be no force, parallel to the surface of the table, between the block and the table; the only force between them would be perpendicular to the table.

Def. When two bodies, which are in contact, are perfectly smooth, the force, or reaction, between them is perpendicular to their common surface at the point of contact.

CHAPTER II,

COMPOSITION AND RESOLUTION OF FORCES

19. SUPPOSE a flat piece of wood is resting on a smooth table, and that it is pulled by means of three strings attached to three of its corners, the forces exerted by the strings being horizontal; if the tensions of the strings be so adjusted that the wood remains at rest, it follows that the three forces are in equilibrium.

Hence two of the forces must together exert a force equal and opposite to the third. This force, equal and opposite to the third, is called the resultant of the first two.

Resultant. Def. *If two or more forces $P, Q, S \dots$ act upon a rigid body, and if a single force, R , can be found whose effect upon the body is the same as that of the forces $P, Q, S \dots$ this single force R is called the resultant of the other forces and the forces $P, Q, S \dots$ are called the components of R .*

20. *Resultant of forces acting in the same straight line.*

If two forces act on a body in the same direction their resultant is clearly equal to their sum; thus two forces acting in the same direction, equal to 5 and 7 lbs. weight respectively, are equivalent to a force of 12 lbs. weight acting in the same direction as the two forces.

If two forces act on a body in opposite directions their resultant is equal to their difference and acts in the direction of the greater; thus two forces acting in opposite directions and equal to 9 and 4 lbs. weight respectively are equivalent to a force of 5 lbs. weight acting in the direction of the first of the two forces.

21. When two forces act at a point of a rigid body in different directions their resultant may be obtained by means of the following

Theorem. Parallelogram of Forces. *If two forces, acting at a point, be represented in magnitude and direction by the two sides of a parallelogram drawn from one of its*

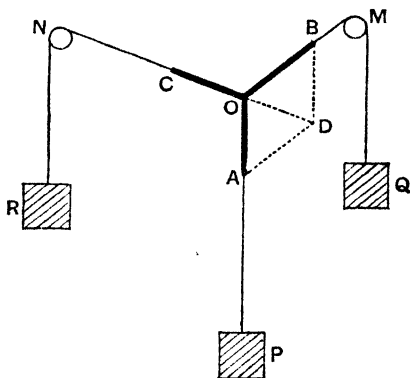
angular points, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram passing through that angular point.

This fundamental theorem of Statics, or rather another form of it, viz. the Triangle of Forces (Art. 30), was first enunciated by Stevinus of Bruges in the year 1586. Before his time the science of Statics rested on the Principle of the Lever as its basis.

In the following article we shall give an experimental proof. In Chapter XVI will be found a proof founded on Newton's Laws of Motion.

22. Experimental proof.

Let M and N be small light smooth pulleys over which pass light strings supporting masses Q lbs. and R lbs. respectively. Let one end of each string be tied at a point O to the end of a third string which supports a mass of P lbs.; then [unless two of the weights are together less than the third] the system will take up some such position as that in the figure. The tension of the string OM is unaltered by passing round the pulley M and is therefore equal to the weight of Q lbs.; so also the tension in the string ON is equal to the weight of R lbs.



Hence the point O is in equilibrium under the action of forces which are equal respectively to the weights of P , Q , and R lbs.

Along the three strings measure off distances OA , OB , and OC proportional to P , Q , and R respectively and complete the parallelogram $OADB$.

Then it will be found that OD is exactly equal in magnitude, and opposite in direction, to OC .

But the effect of the forces OA and OB is equal and opposite to that of OC . Hence the effect of the force OD is exactly the same as that of the forces OA and OB .

This will be found to be true whatever be the relative magnitudes of P , Q , and R , provided only that one of them is not greater than the sum of the other two.

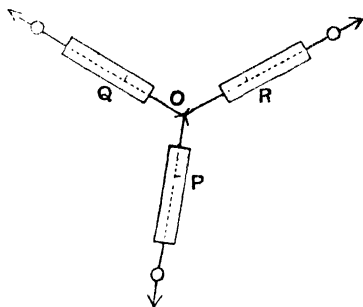
Hence we conclude that the theorem enunciated is always true.

When the experiment is performed, it will probably be found that the point O may be moved into one of several positions close to one another. The reason for this is that we cannot wholly get rid of the friction on the pivots of the pulleys. The effect of this friction will be minimised, in this and similar statical experiments, if the pulleys are of fairly large diameter; aluminium pulleys are suitable because they can be made of comparatively large size and yet be of small weight.

23. The pulleys and weights of the above experiment may be replaced by three Salter's Spring Balances furnished with hooks at their ends as in the annexed figure.

Each of these Balances shews, by a pointer which travels up and down a graduated face, what force is applied to the hook at its end.

The three hooks are fastened together and forces are applied to the rings at the other ends of the instruments and they are allowed to take up their position of equilibrium. The forces, which the pointers denote, replace the



tensions of the strings in the preceding experiment and the rest of the construction follows as before.

24. To find the direction and magnitude of the resultant of two forces, we have thus to find the direction and magnitude of the diagonal of a parallelogram of which the two sides represent the forces.

COMPOSITION AND RESOLUTION OF FORCES 9

Ex. 1. Find the resultant of forces respectively equal to 12 and 5 lbs. weight and acting at right angles.

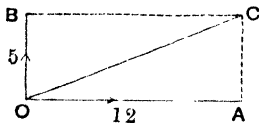
Let OA and OB represent the forces so that OA is 12 units of length and OB is 5 units of length; complete the rectangle $OACB$.

Then

$$OC^2 = OA^2 + AC^2 = 12^2 + 5^2 = 169.$$

$$\therefore OC = 13.$$

$$\text{Also } \tan COA = \frac{AC}{OA} = \frac{5}{12}.$$



Hence the resultant is a force equal to 13 lbs. weight making with the first force an angle whose tangent is $\frac{5}{12}$, i.e. about $22^\circ 37'$.

Ex. 2. Find the resultant of forces equal to the weights of 5 and 3 lbs. respectively acting at an angle of 60° .

Let OA and OB represent the forces, so that OA is 5 units and OB 3 units of length; also let the angle AOB be 60° .

Complete the parallelogram $OACB$ and draw CD perpendicular to OA . Then OC represents the required resultant.

$$\text{Now } AD = AC \cos CAD = 3 \cos 60^\circ = \frac{3}{2}; \therefore OD = \frac{13}{2}.$$

$$\text{Also } DC = AC \sin 60^\circ = \frac{3\sqrt{3}}{2}.$$

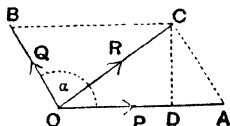
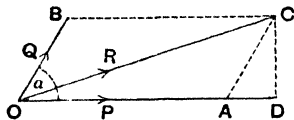
$$\therefore OC = \sqrt{OD^2 + DC^2} = \sqrt{\frac{169}{4} + \frac{27}{4}} = \sqrt{49} = 7,$$

$$\text{and } \tan COD = \frac{DC}{OD} = \frac{3\sqrt{3}}{13} = .3997.$$

Hence the resultant is a force equal to 7 lbs. weight in a direction making with OD an angle whose tangent is .3997, i.e. about $21^\circ 47'$.

25. The resultant, R , of two forces P and Q acting at an angle α may be easily obtained by Trigonometry.

For let OA and OB represent the forces P and Q acting



at an angle α . Complete the parallelogram $OACB$ and draw CD perpendicular to OA , produced if necessary.

Let R denote the magnitude of the resultant.

$$\begin{aligned}\text{Then } OD &= OA + AD = OA + AC \cos DAC \\ &= P + Q \cos BOD = P + Q \cos \alpha.\end{aligned}$$

[If D fall between O and A , as in the second figure, we have
 $OD = OA - DA = OA - AC \cos DAC = P - Q \cos (180^\circ - \alpha) = P + Q \cos \alpha.$]

$$\text{Also } DC = AC \sin DAC = Q \sin \alpha.$$

$$\begin{aligned}\therefore R^2 &= OC^2 = OD^2 + CD^2 = (P + Q \cos \alpha)^2 + (Q \sin \alpha)^2 \\ &= P^2 + Q^2 + 2PQ \cos \alpha.\end{aligned}$$

$$\therefore R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} \dots\dots\dots(i).$$

$$\text{Also } \tan COD = \frac{DC}{OD} = \frac{Q \sin \alpha}{P + Q \cos \alpha} \dots\dots\dots(ii).$$

These two equations give the required magnitude and direction of the resultant.

Cor. 1. If the forces be at right angles, we have $\alpha = 90^\circ$, so that
 $R = \sqrt{P^2 + Q^2}$, and $\tan COA = \frac{Q}{P}$.

- Cor. 2.** (1) When $\alpha = 0$, $R = P + Q$,
 (2) When $\alpha = 30^\circ$, $R = \sqrt{P^2 + Q^2 + \sqrt{3}PQ}$,
 (3) When $\alpha = 45^\circ$, $R = \sqrt{P^2 + Q^2 + \sqrt{2}PQ}$,
 (4) When $\alpha = 60^\circ$, $R = \sqrt{P^2 + Q^2 + PQ}$,
 (5) When $\alpha = 120^\circ$, $R = \sqrt{P^2 + Q^2 - PQ}$,
 (6) When $\alpha = 135^\circ$, $R = \sqrt{P^2 + Q^2 - \sqrt{2}PQ}$,
 (7) When $\alpha = 150^\circ$, $R = \sqrt{P^2 + Q^2 - \sqrt{3}PQ}$,
 (8) When $\alpha = 180^\circ$, $R = P - Q$.

EXAMPLES. I

1. In the following six examples P and Q denote two component forces acting at an angle α and R denotes their resultant.

[The results should also be verified by a graph and accurate measurement.]

- (i) If $P = 24$; $Q = 7$; $\alpha = 90^\circ$; find R .
- (ii) If $P = 13$; $R = 14$; $\alpha = 90^\circ$; find Q .
- (iii) If $P = 7$; $Q = 8$; $\alpha = 60^\circ$; find R .
- (iv) If $P = 5$; $Q = 9$; $\alpha = 120^\circ$; find R .
- (v) If $P = 3$; $Q = 5$; $R = 7$; find α .
- (vi) If $P = 5$; $R = 7$; $\alpha = 60^\circ$; find Q .

2. Find the greatest and least resultants of two forces whose magnitudes are 12 and 8 lbs. weight respectively.

COMPOSITION AND RESOLUTION OF FORCES 11

3. Forces equal respectively to 3, 4, 5, and 6 lbs. weight act on a particle in directions respectively north, south, east, and west; find the direction and magnitude of their resultant.

4. Forces of 84 and 187 lbs. weight act at right angles; find their resultant.

5. Two forces whose magnitudes are P and $P\sqrt{2}$ lbs. weight act on a particle in directions inclined at an angle of 135° to each other; find the magnitude and direction of the resultant.

6. Two forces acting at an angle of 60° have a resultant equal to $2\sqrt{3}$ lbs. weight; if one of the forces be 2 lbs. weight, find the other force.

7. Two equal forces act on a particle; find the angle between them when the square of their resultant is equal to three times their product.

8. Find the magnitude of two forces such that, if they act at right angles, their resultant is $\sqrt{10}$ lbs. weight, whilst, when they act at an angle of 60° , their resultant is $\sqrt{13}$ lbs. weight.

9. Find the angle between two equal forces P when their resultant is equal to P .

10. Two given forces act on a particle; find in what direction a third force of given magnitude must act so that the resultant of the three may be as great as possible.

11. If one of two forces be double the other and the resultant be equal to the greater force, find the cosine of the angle between the forces.

12. Two forces equal to $2P$ and P respectively act on a particle; if the first be doubled and the second be increased by 12 lbs. weight the direction of the resultant is unaltered; find the value of P .

13. The resultant of forces P and Q is R ; if Q be doubled, R is doubled; if Q be reversed, R is again doubled; prove that

$$P : Q : R :: \sqrt{2} : \sqrt{3} : \sqrt{2}.$$

14. By drawing alone solve the following:

- (i) If $P=10$, $Q=15$, and $\alpha=37^\circ$, find R ;
- (ii) If $P=9$, $Q=7$, and $\alpha=133^\circ$, find R ;
- (iii) If $P=7$, $Q=5$, and $R=10$, find α ;
- (iv) If $P=7.3$, $R=8.7$, and $\alpha=65^\circ$, find Q .

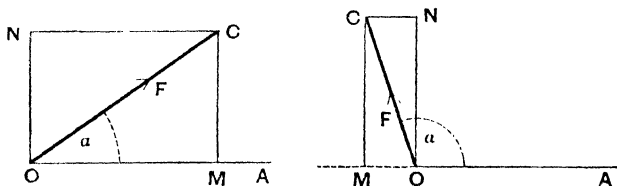
26. Two forces, given in magnitude and direction, have only one resultant; for only one parallelogram can be constructed having two lines OA and OB (Fig. Art. 25) as adjacent sides.

A force may be resolved into two components in an infinite number of ways; for an infinite number of parallelograms can be constructed having OC as a diagonal and each of these parallelograms would give a pair of such components.

27. The most important case of the resolution of forces occurs when we resolve a force into two components **at right angles** to one another.

Suppose we wish to resolve a force F , represented by OC , into two components, one of which is in the direction OA and the other is perpendicular to OA .

Draw CM perpendicular to OA and complete the parallelogram $OMCN$. The forces represented by OM and ON have as their resultant the force OC , so that OM and ON are the required components.



Let the angle $AO C$ be α .

Then $OM = OC \cos \alpha = F \cos \alpha$,

and $ON = MC = OC \sin \alpha = F \sin \alpha$.

[If the point M lie in OA produced backwards, as in the second figure, the component of F in the direction OA

$$= -OM = -OC \cos COM = -OC \cos (180^\circ - \alpha) = OC \cos \alpha = F \cos \alpha.$$

Also the component perpendicular to OA

$$= ON = MC = OC \sin COM = F \sin \alpha.]$$

Hence, in each case, the required components are

$$\mathbf{F \cos \alpha \text{ and } F \sin \alpha.}$$

Thus a force equal to 10 lbs. weight acting at an angle of 60° with the horizontal is equivalent to $10 \cos 60^\circ (= 10 \times \frac{1}{2} = 5$ lbs. weight) in a horizontal direction, and $10 \sin 60^\circ (= 10 \times \frac{\sqrt{3}}{2} = 5 \times 1.732 = 8.66$ lbs. weight) in a vertical direction.

28. Def. The Resolved Part of a given force in a given direction is the component in the given direction which, with a component in a direction perpendicular to the given direction, is equivalent to the given force.

Thus in the previous article the resolved part of the force F in the direction OA is $F \cos \alpha$. Hence

The Resolved Part of a given force in a given direction is obtained by multiplying the given force by the cosine of the angle between the given force and the given direction.

29. A force cannot produce any effect in a direction perpendicular to its own line of action. For (Fig. Art. 27) there is no reason why the force ON should have any tendency to make a particle at O move in the direction OA rather than to make it move in the direction AO produced; hence the force ON cannot have any tendency to make the particle move in either the direction OA or AO produced.

For example, if a railway carriage be standing at rest on a railway line it cannot be made to move along the rails by any force which is acting horizontally and in a direction perpendicular to the rails.

EXAMPLES. II

1. A force equal to 10 lbs. weight is inclined at an angle of 30° to the horizontal; find its resolved parts in a horizontal and vertical direction respectively.

2. Find the resolved part of a force P in a direction making an angle of 45° with its direction.

3. A truck is at rest on a railway line and is pulled by a horizontal force equal to the weight of 100 lbs. in a direction making an angle of 60° with the direction of the rails; what is the force tending to urge the truck forwards?

4. A body, of weight 20 lbs., is placed on an inclined plane whose height is 4 feet and whose length is 5 feet; find the resolved parts of its weight along and perpendicular to the plane.

5. Find a horizontal force and a force inclined at 60° to the vertical whose resultant is a vertical force P .

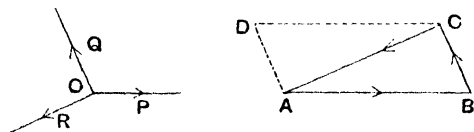
6. By means of drawing, or otherwise, resolve a force of 35 lbs. weight into

(i) two equal forces acting at an angle of 60° ;

(ii) two components making angles of 98° and 40° with it on opposite sides.

30. Triangle of Forces. *If three forces, acting at a point, be represented in magnitude and direction by the sides of a triangle, taken in order, that is, taken the same way round, they will be in equilibrium.*

Let the forces P , Q , and R acting at the point O be represented in magnitude and direction by the sides AB , BC , and CA of the triangle ABC ; they will be in equilibrium.



Complete the parallelogram $ABCD$.

The forces represented by BC and AD are the same, since BC and AD are equal and parallel.

Now the resultant of the forces AB and AD is, by the parallelogram of forces, represented by AC .

Hence the resultant of AB , BC , and CA is equal to the resultant of forces AC and CA , and is therefore zero.

Hence the three forces P , Q , and R are in equilibrium.

Cor. Since forces represented by AB , BC , and CA are in equilibrium, and since, when three forces are in equilibrium, each is equal and opposite to the resultant of the other two, it follows that the resultant of AB and BC is equal and opposite to CA , *i.e.* their resultant is represented by AC .

Hence the resultant of two forces, acting at a point and represented by the sides AB and BC of a triangle, is represented by the third side AC .

31. In the Triangle of Forces the student must carefully note that the forces must be parallel to the sides of a triangle *taken in order*.

For example, if the first force act in the direction AB , the second must act in the direction BC , and the third in the direction CA ; if the second force were in the direction CB , instead of BC , the forces would not be in equilibrium.

The three forces must also act at a point; if the lines of action of the forces were BC , CA , and AB they would not be in equilibrium; for the forces AB and BC would have a resultant, acting at B , equal and parallel to AC . The system of forces would then reduce to two equal and parallel forces acting in opposite directions, and, as we shall see in a later chapter, such a pair of forces could not be in equilibrium.

32. The converse of the Triangle of Forces is also true, *viz.* that *If three forces acting at a point be in equilibrium, they can be represented in magnitude and direction by the sides of any triangle which is drawn so as to have its sides respectively parallel to the directions of the forces.*

Let the three forces P , Q , and R , acting at a point O , be in equilibrium. Measure off lengths OL and OM along the directions of P and Q to represent these forces respectively.

Complete the parallelogram $OLNM$ and join ON .

Since the three forces P , Q , and R are in equilibrium, each must be equal and opposite to the resultant of the other two. Hence R must be equal and opposite to the resultant of P and Q , and must therefore be represented by NO . Also LN is equal and parallel to OM .

Hence the three forces P , Q , and R are parallel and proportional to the sides OL , LN , and NO of the triangle OLN .

Any other triangle, whose sides are parallel to those of the triangle OLN , will have its sides proportional to those of OLN and therefore proportional to the forces.

Again any triangle, whose sides are respectively perpendicular to those of the triangle OLN , will have its sides proportional to the sides of OLN and therefore proportional to the forces.

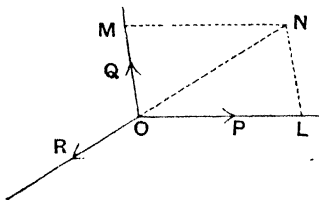
We hence have an easy graphical method to determine the relative directions of three forces which are in equilibrium and whose magnitudes are known. We have to construct a triangle whose sides are proportional to the forces, and this can always be done unless two of the forces added together are less than the third.

33: Lami's Theorem. *If three forces acting on a particle keep it in equilibrium, each is proportional to the sine of the angle between the other two.*

Taking Fig., Art. 32, let the forces P , Q and R be in equilibrium. As before, measure off lengths OL and OM to represent the forces P and Q , and complete the parallelogram $OLNM$. Then NO represents R .

Since the sides of the triangle OLN are proportional to the sines of the opposite angles (Appendix I, Art. 14), we have

$$\frac{OL}{\sin LNO} = \frac{LN}{\sin NOL} = \frac{NO}{\sin OLN}.$$



But $\sin LNO = \sin NOM = \sin (180^\circ - QOR) = \sin QOR$,
 $\sin LON = \sin (180^\circ - POR) = \sin ROP$,
 and $\sin OLN = \sin (180^\circ - POQ) = \sin POQ$.
 Also $LN = OM$.

Hence
$$\frac{OL}{\sin QOR} = \frac{OM}{\sin ROP} = \frac{NO}{\sin POQ},$$

i.e.
$$\frac{P}{\sin QOR} = \frac{Q}{\sin ROP} = \frac{R}{\sin POQ}.$$

Ex. The resultant of two forces acting at an angle of 150° is perpendicular to the smaller of these forces. The greater component being equal to 30 lbs. weight, find the other component and the resultant.

Taking the figure of Art. 32, we have

$$P = 30 \text{ and } POQ = 150^\circ.$$

Also MON is 90° , so that, if R be equal and opposite to the required resultant, then $QOR = 90^\circ$.

Hence Lami's Theorem gives

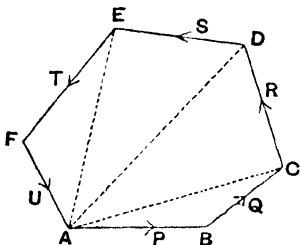
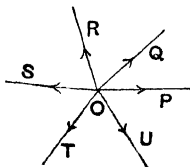
$$\frac{30}{\sin 90^\circ} = \frac{Q}{\sin 120^\circ} = \frac{R}{\sin 150^\circ}, \quad \text{i.e. } 30 = \frac{Q}{\frac{\sqrt{3}}{2}} = \frac{R}{\frac{1}{2}}.$$

$$\therefore Q = 15\sqrt{3} = 25.98 \dots \text{ lbs. wt., and } R = 15 \text{ lbs. wt.}$$

34. Polygon of Forces. If any number of forces, acting on a particle, be represented, in magnitude and direction, by the sides of a polygon, taken in order, the forces shall be in equilibrium.

Let the sides AB , BC , CD , DE , EF , and FA of the polygon $ABCDEF$ represent the forces acting on a particle O . Join AC , AD , and AE .

By the corollary to Art. 30, the resultant of forces AB and BC is represented by AC .



Similarly the resultant of forces AC and CD is represented by AD ; the resultant of forces AD and DE by AE ; and the resultant of forces AE and EF by AF .

Hence the resultant of all the forces is equal to the resultant of AF and FA , i.e. the resultant vanishes.

Hence the forces are in equilibrium.

A similar method of proof will apply whatever be the number of forces. It is also clear from the proof that the sides of the polygon need not be in the same plane.

***35.** The converse of the Polygon of Forces is not true; for the ratios of the sides of a polygon are not known when the directions of the sides are known. For example, in the above figure, we might take any point A' on AB and draw $A'E'$ parallel to AE to meet EF in F' ; the new polygon $A'BCDEF'$ has its sides respectively parallel to those of the polygon $ABCDEF$, but the corresponding sides are clearly not proportional.

***36.** The resultant of two forces, acting at a point O in directions OA and OB and represented in magnitude by $\lambda \cdot OA$ and $\mu \cdot OB$, is represented by $(\lambda + \mu) \cdot OC$, where C is a point in AB such that $\lambda \cdot CA = \mu \cdot CB$.

For let C divide the line AB , such that

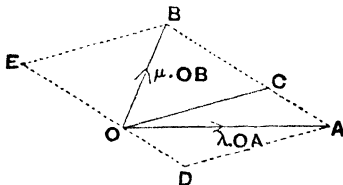
$$\lambda \cdot CA = \mu \cdot CB.$$

Complete the parallelograms $OCAD$ and $OCBE$.

By the parallelogram of forces the force $\lambda \cdot OA$ is equivalent to forces represented by $\lambda \cdot OC$ and $\lambda \cdot OD$.

Also the force $\mu \cdot OB$ is equivalent to forces represented by $\mu \cdot OC$ and $\mu \cdot OE$.

Hence the forces $\lambda \cdot OA$ and $\mu \cdot OB$ are together equivalent to a force $(\lambda + \mu) \cdot OC$ together with forces $\lambda \cdot OD$ and $\mu \cdot OE$.



But (since $\lambda \cdot OD = \lambda \cdot CA = \mu \cdot CB = \mu \cdot OE$) these two latter forces are equal and opposite and therefore are in equilibrium.

Hence the resultant is $(\lambda + \mu) \cdot OC$.

Cor. The resultant of forces represented by OA and OB is $2OC$, where C is the middle point of AB .

This is also clear from the fact that OC is half the diagonal OD of the parallelogram of which OA and OB are adjacent sides.

EXAMPLES. III

1. Three forces acting at a point are in equilibrium; if they make angles of 120° with one another, shew that they are equal.
If the angles are 60° , 150° , and 150° , in what proportions are the forces?
2. Three forces acting on a particle are in equilibrium; the angle between the first and second is 90° and that between the second and third is 120° ; find the ratios of the forces.
3. Forces equal to $7P$, $5P$, and $8P$ acting on a particle are in equilibrium; find the angle between the latter pair of forces.
4. Two forces act at an angle of 120° . The greater is represented by 80 , and the resultant is at right angles to the less. Find the latter.
5. Two forces acting on a particle are at right angles, and are balanced by a third force making an angle of 150° with one of them. The greater of the two forces being 3 lbs. weight, what must be the values of the others?
6. The magnitudes of two forces are as 3 : 5, and the direction of the resultant is at right angles to that of the smaller force; compare the magnitudes of the greater force and the resultant.
7. The sum of two forces is 18, and the resultant, whose direction is perpendicular to the lesser of the two forces, is 12; find the magnitudes of the forces.
8. If two forces P and Q act at such an angle that $R=P$, shew that, if P be doubled, the new resultant is at right angles to Q .
9. The resultant of two forces P and Q is equal to $Q\sqrt{3}$ and makes an angle of 30° with the direction of P ; prove that P is either equal to, or double of, Q .
10. Construct geometrically the directions of two forces $2P$ and $3P$ which make equilibrium with a force of $4P$ whose direction is given.
11. The sides AB and AC of a triangle ABC are bisected in D and E ; shew that the resultant of forces represented by BE and DC is represented in magnitude and direction by $\frac{3}{2}BC$.
12. P is a particle acted on by forces represented by $\lambda \cdot AP$ and $\lambda \cdot PB$ where A and B are two fixed points; shew that their resultant is constant in magnitude and direction wherever the point P may be.
13. $ABCD$ is a parallelogram; a particle P is attracted towards A and C by forces which are proportional to PA and PC respectively and repelled from B and D by forces proportional to PB and PD ; shew that P is in equilibrium wherever it is situated.

COMPOSITION AND RESOLUTION OF FORCES 19

The following are to be solved by **geometric construction and measurement**. In each case P and Q are two forces inclined at an angle α , and R is their resultant making an angle θ with P .

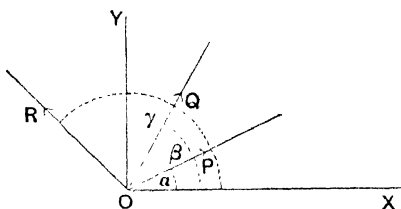
14. $P=50$ kilog., $Q=60$ kilog. and $R=70$ kilog.; find α and θ .
15. $P=30$, $R=40$ and $\alpha=130^\circ$; find Q and θ .
16. $P=60$, $\alpha=75^\circ$ and $\theta=40^\circ$; find Q and R .
17. $P=60$, $R=40$ and $\theta=50^\circ$; find Q and α .
18. $P=80$, $\alpha=55^\circ$ and $R=100$; find Q and θ .
19. $P=25$ lbs. wt., $Q=20$ lbs. wt. and $\theta=35^\circ$; find R and α .
20. A boat is being towed by means of a rope which makes an angle of 20° with the boat's length; assuming that the resultant reaction R of the water on the boat is inclined at 40° to the boat's length and that the tension of the rope is 5 cwt., find, by drawing, the resultant force on the boat, supposing it to be in the direction of the boat's length, and also the value of R .

CHAPTER III

COMPOSITION AND RESOLUTION OF FORCES (*continued*)

37. To find the resultant of any number of forces in one plane acting upon a particle.

Let the forces $P, Q, R \dots$ act upon a particle at O .



Through O draw a fixed line OX and a line OY at right angles to OX .

Let the forces $P, Q, R \dots$ make angles $\alpha, \beta, \gamma \dots$ with OX .

The components of the force P in the directions OX and OY are, by Art. 27, $P \cos \alpha$ and $P \sin \alpha$ respectively; similarly, the components of Q are $Q \cos \beta$ and $Q \sin \beta$; and so for the other forces.

Hence the forces are equivalent to a component,

$$P \cos \alpha + Q \cos \beta + R \cos \gamma \dots \text{ along } OX,$$

and a component,

$$P \sin \alpha + Q \sin \beta + R \sin \gamma \dots \text{ along } OY.$$

Let these components be X and Y respectively, and let F be their resultant inclined at an angle θ to OX .

Since F is equivalent to $F \cos \theta$ along OX , and $F \sin \theta$ along OY , we have

$$F \cos \theta = X \dots\dots\dots(1),$$

$$F \sin \theta = Y \dots\dots\dots(2).$$

Hence, by squaring and adding,

$$F^2 = X^2 + Y^2.$$

Also, by division, $\tan \theta = \frac{Y}{X}$.

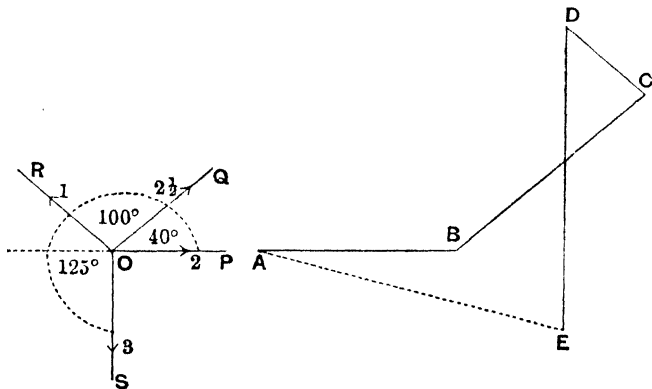
These two equations give F and θ , i.e. the magnitude and direction of the required resultant.

38. Graphical Construction. The resultant of a system of forces acting at a point may also be obtained by means of the Polygon of Forces. For (Fig. Art. 34) forces acting at a point O and represented in magnitude and direction by the sides of the polygon $ABCDEF$ are in equilibrium. Hence the resultant of forces represented by AB , BC , CD , DE , and EF must be equal and opposite to the remaining force FA , i.e. the resultant must be represented by AF .

It follows that the resultant of forces P , Q , R , S , and T acting on a particle may be obtained thus; take a point A and draw AB parallel and proportional to P , and in succession BC , CD , DE , and EF parallel and proportional respectively to Q , R , S , and T ; the required resultant will be represented in magnitude and direction by the line AF .

The same construction would clearly apply for any number of forces.

Ex. Four forces equal to 2, $2\frac{1}{2}$, 1, and 3 kilogrammes wt. act along straight lines OP , OQ , OR , and OS , such that $\angle POQ = 40^\circ$, $\angle QOR = 100^\circ$, and $\angle ROS = 125^\circ$; find their resultant in magnitude and direction.



Draw AB parallel to OP and equal to 2 inches; through B draw BC parallel to OQ and equal to 2.5 inches, and then CD parallel to OR and

equal to 1 inch, and finally DE parallel to OS and equal to 3 inches. On measurement we have AE equal to 2.99 inches and the angle BAE equal to a little over 14° .

Hence the resultant is 2.99 kilogrammes wt. acting at 14° to OP .

39. Ex. 1. *A particle is acted upon by three forces, in one plane, equal to 2, $2\sqrt{2}$, and 1 lbs. weight respectively; the first is horizontal, the second acts at 45° to the horizontal, and the third is vertical; find their resultant.*

Here $X = 2 + 2\sqrt{2} \cos 45^\circ + 0 = 2 + 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 4,$

$$Y = 0 + 2\sqrt{2} \sin 45^\circ + 1 = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1 = 3.$$

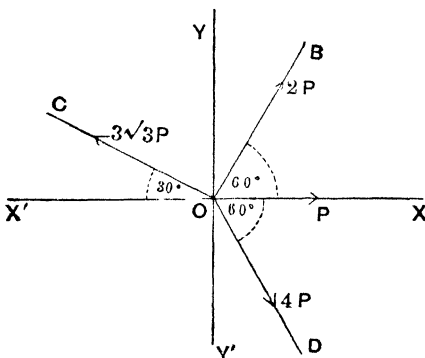
Hence $F \cos \theta = 4,$ and $F \sin \theta = 3;$

$$\therefore F = \sqrt{4^2 + 3^2} = 5, \text{ and } \tan \theta = \frac{3}{4}.$$

The resultant is therefore a force equal to 5 lbs. weight acting at an angle with the horizontal whose tangent is $\frac{3}{4}$, i.e. $36^\circ 52'.$

Ex. 2. *A particle is acted upon by forces represented by $P, 2P, 3\sqrt{3}P,$ and $4P$; the angles between the first and second, the second and third, and the third and fourth are $60^\circ, 90^\circ,$ and 150° respectively. Shew that the resultant is a force P in a direction inclined at an angle of 120° to that of the first force.*

In this example it will be a simplification if we take the fixed



line OX to coincide with the direction of the first force P ; let OXO' and YOY' be the two fixed lines at right angles.

The second, third, and fourth forces are respectively in the first, second, and fourth quadrants, and we have clearly

$$BOX = 60^\circ, COX' = 30^\circ, \text{ and } DOX = 60^\circ.$$

The first force has no component along OY .

The second force is equivalent to components $2P \cos 60^\circ$ and $2P \sin 60^\circ$ along OX and OY respectively.

The third force is equivalent to forces

$$3\sqrt{3}P \cos 30^\circ \text{ and } 3\sqrt{3}P \sin 30^\circ$$

along OX' and OY respectively, *i.e.* to forces $-3\sqrt{3}P \cos 30^\circ$ and $3\sqrt{3}P \sin 30^\circ$ along OX and OY .

So the fourth force is equivalent to $4P \cos 60^\circ$ and $4P \sin 60^\circ$ along OX and OY' , *i.e.* to $4P \cos 60^\circ$ and $-4P \sin 60^\circ$ along OX and OY .

Hence $X = P + 2P \cos 60^\circ - 3\sqrt{3}P \cos 30^\circ + 4P \cos 60^\circ$

$$= P + P - \frac{9P}{2} + 2P = -\frac{P}{2},$$

and $Y = 0 + 2P \sin 60^\circ + 3\sqrt{3}P \sin 30^\circ - 4P \sin 60^\circ$

$$= P\sqrt{3} + \frac{3\sqrt{3}}{2}P - 4P \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}P.$$

Hence, if F be the resultant at an angle θ with OX , we have

$$F = \sqrt{X^2 + Y^2} = P,$$

and

$$\tan \theta = \frac{Y}{X} = -\sqrt{3} = \tan 120^\circ,$$

so that the resultant is a force P at an angle of 120° with the first force.

EXAMPLES. IV

[Exs. 2, 3, 4, 5, and 8 are suitable for **graphic** solutions.]

1. Forces of 1, 2, and $\sqrt{3}$ lbs. weight act at a point A in directions AP , AQ , and AR , the angle PAQ being 60° and PAR a right angle; find their resultant.

2. A particle is acted on by forces of 5 and 3 lbs. weight which are at right angles and by a force of 4 lbs. weight bisecting the angle between them; find the magnitude of the force that will keep it at rest.

3. Three equal forces, P , diverge from a point, the middle one being inclined at an angle of 60° to each of the others. Find the resultant of the three.

4. Three forces $5P$, $10P$, and $13P$ act in one plane on a particle, the angle between any two of their directions being 120° . Find the magnitude of their resultant.

5. Forces $2P$, $3P$, and $4P$ act at a point in directions parallel to the sides of an equilateral triangle taken in order; find the magnitude and line of action of the resultant.

6. Two forces equal respectively to 9 and 12 lbs. weight act at an angle of 135° on a particle; a third force, equal to 10 lbs. weight, acts on the particle, its direction being between the first two and at 30° to the first force; find the magnitude of the resultant of these forces.

7. $ABCD$ is a square; forces of 1 lb. wt., 6 lbs. wt., and 9 lbs. wt. act in the directions AB , AC , and AD respectively; find the magnitude of their resultant correct to two places of decimals.

8. Five forces, acting at a point, are in equilibrium; four of them, whose respective magnitudes are 4, 4, 1, and 3 lbs. weight, make, in succession, angles of 60° with one another. Find the magnitude of the fifth force.

By a graphic construction, or by the use of trigonometrical Tables, find the magnitude, to two places of decimals, and direction, to the nearest degree, of the resultant of

9. three forces equal to 11, 7, and 8 lbs. weight making angles of $18^\circ 18'$, $74^\circ 50'$, and $130^\circ 20'$ with a fixed line.

10. four forces equal to 4, 3, 2, and 1 lbs. weight making angles of 20° , 40° , 60° , and 80° with a fixed line.

11. four forces equal to 8, 12, 15, and 20 lbs. weight making angles of 30° , 70° , $120^\circ 15'$, and 155° with a fixed line.

12. three forces equal to 85, 47, and 63 kilogs. weight acting along lines OA , OB , and OC , where $\angle AOB = 78^\circ$ and $\angle BOC = 125^\circ$.

13. A boat is towed along a river by means of two ropes, attached to the same point, which are pulled by two men who keep at opposite points of the bank 50 feet apart; one rope is 30 feet long and is pulled with a force equal to the weight of 35 lbs., and the other rope is 45 feet long; the boat is in this way made to move uniformly in a straight line; find, by a graphic construction, the resistance offered to the boat by the stream and the tension of the second rope.

40. To find the conditions of equilibrium of any number of forces acting upon a particle.

Let the forces act upon a particle O as in Art. 37.

If the forces balance one another the resultant must vanish, i.e. F must be zero.

Hence $X^2 + Y^2 = 0$.

Now the sum of the squares of two real quantities cannot be zero unless each quantity is separately zero;

$$\therefore X = 0, \text{ and } Y = 0.$$

Hence, if the forces acting on a particle are in equilibrium, the algebraic sum of their resolved parts in each of two directions at right angles is zero.

Conversely, if the sum of their resolved parts in each of two directions at right angles vanishes, the forces are in equilibrium.

For, in this case, both X and Y are zero, and therefore F is zero also.

Hence, since the resultant of the forces vanishes, the forces are in equilibrium.

41. When there are only three forces acting on a particle the conditions of equilibrium are often most easily found by applying Lami's Theorem (Art. 33).

42. Ex. 1. A body of 65 lbs. weight is suspended by two strings of lengths 5 and 12 feet attached to two points in the same horizontal line whose distance apart is 13 feet; find the tensions of the strings.

Let AC and BC be the two strings, so that

$AC = 5$ ft., $BC = 12$ ft., and $AB = 13$ ft.

Since $13^2 = 12^2 + 5^2$, the angle ACB is a right angle.

Let the direction CE of the weight be produced to meet AB in D ; also let the angle CBA be θ , so that

$$\angle ACD = 90^\circ - \angle BCD = \angle CBD = \theta.$$

Let T_1 and T_2 be the tensions of the strings. By Lami's Theorem we have

$$\frac{T_1}{\sin ECB} = \frac{T_2}{\sin ECA} = \frac{65}{\sin ACB};$$

$$\therefore \frac{T_1}{\sin BCD} = \frac{T_2}{\sin \theta} = \frac{65}{\sin 90^\circ};$$

$$\therefore T_1 = 65 \cos \theta, \text{ and } T_2 = 65 \sin \theta.$$

But $\cos \theta = \frac{BC}{BA} = \frac{12}{13}$, and $\sin \theta = \frac{AC}{AB} = \frac{5}{13}$;

$$\therefore T_1 = 60, \text{ and } T_2 = 25 \text{ lbs. wt.}$$

Otherwise thus: The triangle ACB has its sides respectively perpendicular to the directions of the forces T_1 , T_2 , and 65;

$$\therefore \frac{T_1}{BC} = \frac{T_2}{CA} = \frac{65}{AB};$$

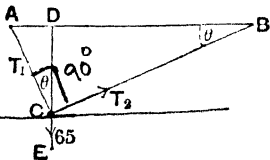
$$\therefore T_1 = 65 \frac{BC}{AB} = 60, \text{ and } T_2 = 65 \frac{AC}{AB} = 25.$$

Or again, we may apply the result of Art. 40. Equating to zero the sum of the resolved parts in the horizontal and vertical directions, we have

$$T_2 \cos CBA - T_1 \cos CAB = 0,$$

and $T_2 \sin CBA + T_1 \sin CAB - 65 = 0.$

But $\cos CBA = \frac{CB}{AB} = \frac{12}{13}$, and $\sin CBA = \frac{CA}{AB} = \frac{5}{13}.$



Also $\cos CAB = \frac{5}{13}$, and $\sin CAB = \frac{12}{13}$.

The above equations therefore become

$$12T_2 - 5T_1 = 0,$$

and

$$5T_2 + 12T_1 = 65 \times 13.$$

Solving, we have $T_1 = 60$ and $T_2 = 25$.

EXAMPLES. V

1. Two men carry a weight W between them by means of two ropes fixed to the weight; one rope is inclined at 45° to the vertical and the other at 30° ; find the tension of each rope.

$$\left[\sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}} \right]$$

2. A body, of mass 2 lbs., is fastened to a fixed point by means of a string of length 25 inches; it is acted on by a horizontal force F and rests at a distance of 20 inches from the vertical line through the fixed point; find the value of F and the tension of the string.

3. A body, of mass 130 lbs., is suspended from a horizontal beam by strings, whose lengths are respectively 1 ft. 4 ins. and 5 ft. 3 ins., the strings being fastened to the beam at two points 5 ft. 5 ins. apart. What are the tensions of the strings?

4. A body, of mass 70 lbs., is suspended by strings, whose lengths are 6 and 8 feet respectively, from two points in a horizontal line whose distance apart is 10 feet; find the tensions of the strings.

5. A mass of 60 lbs. is suspended by two strings of lengths 9 and 12 feet respectively, the other ends of the strings being attached to two points in a horizontal line at a distance of 15 feet apart; find the tensions of the strings. Use the $\perp$ formula

6. A string suspended from a ceiling supports three bodies, each of mass 4 lbs., one at its lowest point and each of the others at equal distances from its extremities; find the tensions of the parts into which the string is divided.

7. Two equal masses, of weight W , are attached to the extremities of a thin string which passes over 3 tacks in a wall arranged in the form of an isosceles triangle, with the base horizontal and with a vertical angle of 120° ; find the thrust on each tack.

8. A stream is 96 feet wide and a boat is dragged down the middle of the stream by two men on opposite banks, each of whom pulls with a force equal to 100 lbs. wt.; if the ropes be attached to the same point of the boat and each be of length 60 feet, find the resultant force on the boat.

9. Two masses, each equal to 112 lbs., are joined by a string which passes over two small smooth pegs, A and B , in the same horizontal plane; if a mass of 5 lbs. be attached to the string halfway between

A and B , find in inches the depth to which it will descend below the level of AB , supposing AB to be 10 feet.

What would happen if the small mass were attached to any other point of the string?

10. A heavy chain has weights of 10 and 16 lbs. attached to its ends and hangs in equilibrium over a smooth pulley; if the greatest tension of the chain be 20 lbs. wt., find the weight of the chain.

11. A heavy chain, of length 8 ft. 9 ins. and weighing 15 lbs., has a weight of 7 lbs. attached to one end and is in equilibrium hanging over a smooth peg. What length of the chain is on each side?

12. A body is free to slide on a smooth vertical circular wire and is connected by a string, equal in length to the radius of the circle, to the highest point of the circle; find the tension of the string and the reaction of the circle.

13. Explain how the force of the current may be used to urge a ferry-boat across the river, assuming that the centre of the boat is attached by a long rope to a fixed point in the middle of the stream.

14. Explain how a ship is enabled to sail in a direction nearly opposite to that of the wind.

[Let AB be the direction of the keel and therefore that of the ship's motion, and OA the apparent direction of the wind, the angle OAB being acute and equal to α . Let AC be the direction of the sail, AC being between AB and AO , and the angle BAC being θ .

Let P be the force exerted by the wind in a direction perpendicular to the sail. Resolve it into two components, $P \cos \theta$ perpendicular to AB and $P \sin \theta$ along AB . The former component produces leeway (*i.e.* motion sideways). The latter is never zero unless θ or P vanishes. Also P never entirely vanishes unless the direction of the wind coincides with that of the sail.]

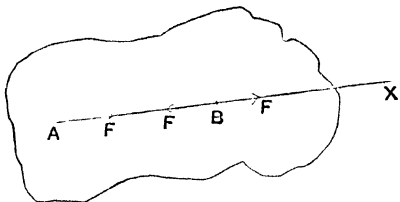
CHAPTER IV

PARALLEL FORCES

43. *Introduction, or removal, of equal and opposite forces.* We shall assume that if at any point of a rigid body we apply two equal and opposite forces, they will have no effect on the equilibrium of the body; similarly, that if at any point of a body two equal and opposite forces are acting they may be removed.

44. *Principle of the Transmissibility of Force.* If a force act at any point of a rigid body, it may be considered to act at any other point in its line of action provided that this latter point is rigidly connected with the body.

Let a force F act at a point A of a body in a direction AX . Take any point B in AX and at B introduce two equal and opposite forces, each equal to F , acting in the directions BA and BX ; these will have no effect on the equilibrium of the body.



The forces F acting at A in the direction AB , and F at B in the direction BA , are equal and opposite; we shall assume that they neutralise one another and hence that they may be removed.

We have thus left the force F at B acting in the direction BX , and its effect is the same as that of the original force F at A .

The internal forces in the above body would be different according as the force F is supposed applied at A or B ; of the internal forces, however, we do not treat in the present book.

45. In Chapters II and III we have shewn how to find the resultant of forces which meet in a point. In the present chapter we shall consider the composition of parallel forces.

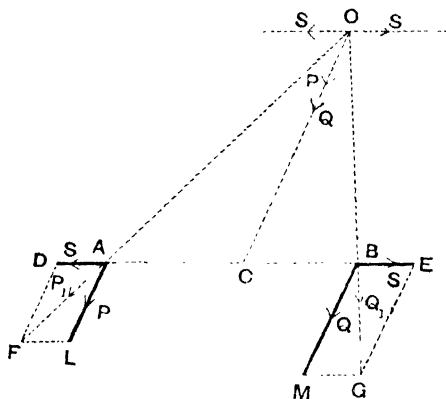
In the ordinary statical problems of every-day life parallel forces are of constant occurrence.

46. Def. Two parallel forces are said to be **like** when they act in the same direction; when they act in opposite parallel directions they are said to be **unlike**.

47. To find the resultant of two parallel forces acting upon a rigid body.

Case I. Let the forces be like.

Let P and Q be the forces acting at points A and B of the body, and let them be represented by the lines AL and BM .



Join AB , and at A and B apply two equal and opposite forces, each equal to S , and acting in the directions BA and AB respectively. Let these forces be represented by AD and BE . These two forces balance one another and have no effect upon the equilibrium of the body.

Complete the parallelograms $ALFD$ and $BMGE$; let the diagonals FA and GB be produced to meet in O . Draw OC parallel to AL or BM to meet AB in C .

The forces P and S at A have a resultant P_1 , represented by AF . Let its point of application be removed to O .

So the forces Q and S at B have a resultant Q_1 represented by BG . Let its point of application be transferred to O .

The force P_1 at O may be resolved into two forces, S parallel to AD , and P in the direction OC .

So the force Q_1 at O may be resolved into two forces, S parallel to BE , and Q in the direction OC .

Also these two forces S acting at O are in equilibrium.

Hence the original forces P and Q are equivalent to a force $(P + Q)$ acting along OC , i.e. acting at C parallel to the original directions of P and Q .

To determine the position of the point C . The triangle OCA is, by construction, equiangular with the triangle ALF ;

$$\therefore \frac{OC}{CA} = \frac{AL}{LF} = \frac{P}{S} \text{ (Appendix I, Art. 2).....(1).}$$

So, since the triangles OCB and BMG are equiangular, we have

$$\frac{OC}{CB} = \frac{BM}{MG} = \frac{Q}{S} \dots\dots\dots (2).$$

Hence, from (1) and (2), by division,

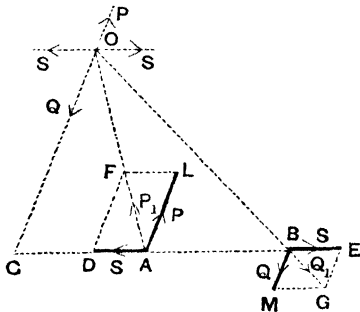
$$\frac{CA}{CB} = \frac{Q}{P},$$

i.e. C divides the line AB internally in the inverse ratio of the forces.

Case II. *Let the forces be unlike.*

Let P and Q be the forces (P being the greater) acting at points A and B of the body, and let them be represented by the lines AL and BM .

Join AB , and at A and B apply two equal and opposite forces, each equal to S , and acting in the directions BA and AB respectively. Let these forces be represented by AD and BE respectively; they balance one another and have no effect on the equilibrium of the body.



Complete the parallelograms $ALFD$ and $BMGE$, and produce the diagonals AF and GB to meet in O .

[These diagonals will always meet unless they be parallel, in which case the forces P and Q will be equal.]

Draw OC parallel to AL or BM to meet AB in C .

The forces P and S acting at A have a resultant P_1 represented by AF . Let its point of application be transferred to O .

So the forces Q and S acting at B have a resultant Q_1 represented by BG . Let its point of application be transferred to O .

The force P_1 at O may be resolved into two forces, S parallel to AD , and P in the direction CO produced.

So the force Q_1 at O may be resolved into two forces, S parallel to BE , and Q in the direction OC .

Also these two forces S acting at O are in equilibrium.

Hence the original forces P and Q are equivalent to a force $(P - Q)$ acting in the direction CO produced, i.e. acting at C in a direction parallel to that of P .

To determine the position of the point C . The triangle OCA is, by construction, equiangular with the triangle FDA ;

$$\therefore \frac{OC}{CA} = \frac{FD}{DA} = \frac{AL}{AD} = \frac{P}{S} \text{ (Appendix I, Art. 2) } \dots(1).$$

So, since the triangles OCB and BMG are equiangular, we have

$$\frac{OC}{CB} = \frac{BM}{MG} = \frac{Q}{S} \dots\dots\dots(2).$$

Hence, from (1) and (2), by division, $\frac{CA}{CB} = \frac{Q}{P}$, i.e. C divides the line AB externally in the inverse ratio of the forces.

To sum up; If two parallel forces, P and Q , act at points A and B of a rigid body,

(i) their resultant is a force whose line of action is parallel to the lines of action of the component forces; also, when the component forces are like, its direction is the same as that of the two forces, and, when the forces are unlike, its direction is the same as that of the greater component;

(ii) the point of application is a point C in AB such that

$$P \cdot AC = Q \cdot BC;$$

(iii) the magnitude of the resultant is the sum of the two component forces when the forces are like, and the difference of the two component forces when they are unlike.

48. *Case of failure of the preceding construction.*

In the second figure of the last article, if the forces P and Q are equal, the triangles FDA and GEB are equal in all respects, and hence the angles DAF and EBG will be equal.

In this case the lines AF and GB will be parallel and will not meet in any such point as O ; hence the construction fails.

Hence there is no single force which is equivalent to two equal unlike parallel forces.

We shall return to the consideration of this case in Chapter vi.

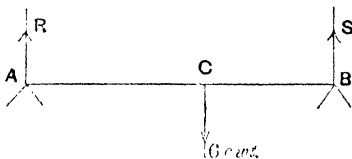
49. If we have a number of like parallel forces acting on a rigid body we can find their resultant by successive applications of Art. 47. We must find the resultant of the first and second, and then the resultant of this resultant and the third, and so on.

The magnitude of the final resultant is the sum of the forces.

If the parallel forces be not all like, the magnitude of the resultant will be found to be the algebraic sum of the forces each with its proper sign prefixed.

50. Ex. *A horizontal rod, 6 feet long, whose weight may be neglected, rests on two supports at its extremities; a body, of weight 6 cwt., is suspended from the rod at a distance of $2\frac{1}{2}$ feet from one end; find the reaction at each point of support. If one support could only bear a pressure equal to the weight of 1 cwt., what is the greatest distance from the other support at which the body could be suspended?*

Let AB be the rod and R and S the pressures at the points of support. Let C be the point at which the body is suspended so that $AC = 3\frac{1}{2}$ and $CB = 2\frac{1}{2}$ feet. For equilibrium the resultant of R and S must balance 6 cwt. Hence, by Art. 47,



$$R + S = 6 \dots\dots\dots(1),$$

and

$$\frac{R}{S} = \frac{BC}{AC} = \frac{2\frac{1}{2}}{3\frac{1}{2}} = \frac{5}{7} \dots\dots\dots(2).$$

Solving (1) and (2), we have $R = \frac{5}{2}$, and $S = \frac{7}{2}$. Hence the pressures are $2\frac{1}{2}$ and $3\frac{1}{2}$ cwt. respectively.

If the reaction at A can only be equal to 1 cwt., S must be 5 cwt. Hence, if AC be x , we have

$$\frac{1}{5} = \frac{RC}{AC} = \frac{6-x}{x}.$$

$$\therefore x = 5 \text{ feet.}$$

Hence BC is one foot.

EXAMPLES. VI

In the four following examples A and B denote the points of application of parallel forces P and Q , and C is the point in which their resultant R meets AB .

1. Find the magnitude and position of the resultant (the forces being like) when

(i) $P=4$; $Q=7$; $AB=11$ inches;

(ii) $P=11$; $Q=19$; $AB=2\frac{1}{2}$ feet;

(iii) $P=5$; $Q=5$; $AB=3$ feet.

2. Find the magnitude and position of the resultant (the forces being unlike) when

(i) $P=17$; $Q=25$; $AB=8$ inches;

(ii) $P=23$; $Q=15$; $AB=40$ inches;

(iii) $P=26$; $Q=9$; $AB=3$ feet.

3. The forces being like,

(i) if $P=8$; $R=17$; $AC=4\frac{1}{2}$ inches; find Q and AB ;

(ii) if $Q=11$; $AC=7$ inches; $AB=8\frac{3}{4}$ inches; find P and R ;

(iii) if $P=6$; $AC=9$ inches; $CB=8$ inches; find Q and R .

4. The forces being unlike,

(i) if $P=8$; $R=17$; $AC=4\frac{1}{2}$ inches; find Q and AB ;

(ii) if $Q=11$; $AC=-7$ inches; $AB=8\frac{3}{4}$ inches; find P and R ;

(iii) if $P=6$; $AC=-9$ inches; $AB=12$ inches; find Q and R .

5. Find two like parallel forces acting at a distance of 2 feet apart, which are equivalent to a given force of 20 lbs. wt., the line of action of one being at a distance of 6 inches from the given force.

6. Find two unlike parallel forces acting at a distance of 18 inches apart, which are equivalent to a force of 30 lbs. wt., the greater of the two forces being at a distance of 8 inches from the given force.

7. Two men carry a heavy cask of weight $1\frac{1}{2}$ cwt., which hangs from a light pole, of length 6 feet, each end of which rests on a shoulder of one of the men. The point from which the cask is hung is one foot nearer to one man than to the other. What is the thrust on each shoulder?

8. Two men, one stronger than the other, have to remove a block of stone weighing 270 lbs. by means of a light plank whose length is 6 feet; the stronger man is able to carry 180 lbs.; how must the block be placed so as to allow him that share of the weight?

9. A uniform rod, 12 feet long and weighing 17 lbs., can turn freely about a point in it and the rod is in equilibrium when a weight of 7 lbs. is hung at one end; how far from the end is the point about which it can turn?

N.B. *The weight of a uniform rod may be taken to act at its middle point.*

10. A straight uniform rod is 3 feet long; when a load of 5 lbs. is placed at one end it balances about a point 3 inches from that end; find the weight of the rod.

11. A uniform bar, of weight 3 lbs. and length 4 feet, passes over a prop and is supported in a horizontal position by a force equal to 1 lb. wt. acting vertically upwards at the other end; find the distance of the prop from the centre of the bar.

12. A heavy uniform rod, 4 feet long, rests horizontally on two pegs which are one foot apart; a weight of 10 lbs. suspended from one end, or a weight of 4 lbs. suspended from the other end, will just tilt the rod up; find the weight of the rod and the distances of the pegs from the centre of the rod.

13. A uniform iron rod, $2\frac{1}{2}$ feet long and of weight 8 lbs., is placed on two rails fixed at two points, *A* and *B*, in a vertical wall. *AB* is horizontal and 5 inches long; find the distances at which the ends of the rod extend beyond the rails if the difference of the reactions of the rails is 6 lbs. wt.

14. A uniform beam, 4 feet long, is supported in a horizontal position by two props, which are 3 feet apart, so that the beam projects one foot beyond one of the props; shew that the reaction of one prop is double that of the other.

15. One end of a heavy uniform rod, of weight *W*, rests on a smooth horizontal plane, and a string tied to the other end of the rod is fastened to a fixed point above the plane; find the tension of the string.

16. A man carries a weight of 50 lbs. at the end of a stick, 3 feet long, resting on his shoulder. He regulates the stick so that the length between his shoulder and his hands is (1) 12, (2) 18, and (3) 24 inches; how great are the forces exerted by his hand and the pressures on his shoulder in each case?

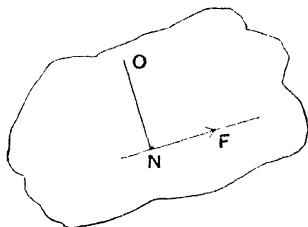
17. A portmanteau, of length 3 feet and height 2 feet and whose centre of gravity is at its centre of figure, is carried upstairs by two men who hold it by the front and back edges of its lower face. If this face is inclined at 30° to the horizontal and the weight of the portmanteau is one cwt., find how much of the weight each man supports.

CHAPTER V

MOMENTS

51. Def. The moment of a force about a given point is the product of the force and the perpendicular drawn from the given point upon the line of action of the force.

Thus the moment of a force F about a given point O is $F \times ON$, where ON is the perpendicular drawn from O upon the line of action of F .

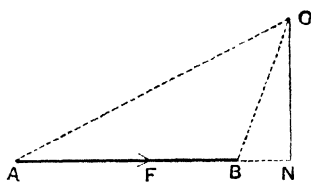
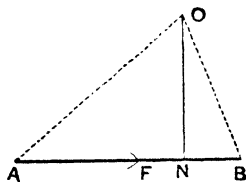


It will be noted that the moment of a force F about a given point O never vanishes, unless, either the force vanishes, or the force passes through the point about which the moment is taken.

52. Geometrical representation of a moment.

Suppose the force F to be represented in magnitude, direction, and line of action by the line AB . Let O be any given point and ON the perpendicular from O upon AB or AB produced.

Join OA and OB .

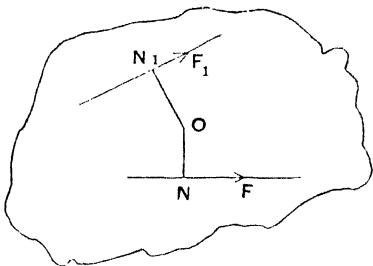


By definition the moment of F about O is $F \times ON$, i.e. $AB \times ON$. But $AB \times ON$ is equal to twice the area of the triangle OAB [for it is equal to the area of a rectangle whose

base is AB and whose height is ON]. Hence the moment of the force F about the point O is represented by twice the area of the triangle OAB , i.e. by twice the area of the triangle whose base is the line representing the force and whose vertex is the point about which the moment is taken.

53. Physical meaning of the moment of a force about a point.

Suppose the body in the figure of Art. 51 to be a plane lamina [i.e. a body of very small thickness, such as a piece of sheet-tin or a thin piece of board] resting on a smooth table and suppose the point O of the body to be fixed. The effect of a force F acting on the body would be to cause it to turn about the point O as a centre, and this effect would not be zero unless (1) the force F were zero, or (2) the force F passed through O , in which case the distance ON would vanish. Hence the product $F \times ON$ would seem to be a fitting measure of the tendency of F to turn the body about O . This may be experimentally verified as follows:



Let the lamina be at rest under the action of two forces F and F_1 , whose lines of action lie in the plane of the lamina. Let ON and ON_1 be the perpendiculars drawn from the fixed point O upon the lines of action of F and F_1 .

If we measure the lengths ON and ON_1 and also the forces F and F_1 , it will be found that the product $F \cdot ON$ is always equal to the product $F_1 \cdot ON_1$.

Hence the two forces, F and F_1 , will have equal but opposite tendencies to turn the body about O if their moments about O have the same magnitude.

These forces F and F_1 may be measured by carrying the strings over light smooth pulleys and hanging weights at their ends sufficient to give equilibrium; or by tying the strings to the hooks of two spring balances and noting

the readings of the balances, as in the cases of Arts. 22 and 23.

54. Positive and negative moments. In Art. 53 the force F would, if it were the only force acting on the lamina, make it turn in a direction opposite to that in which the hands of a watch move, when the watch is laid on the table with its face upwards.

The force F_1 would, if it were the only force acting on the lamina, make it turn in the same direction as that in which the hands of the watch move.

The moment of F about O is said to be *positive*, and the moment of F_1 about O is said to be *negative*.

55. Algebraic sum of moments. The algebraic sum of the moments of a set of forces about a given point is the sum of the moments of the forces, each moment having its proper sign prefixed to it.

Ex. $ABCD$ is a square; along the sides AB , CB , DC , and DA forces act equal respectively to 6, 5, 8, and 12 lbs. wt. Find the algebraic sum of their moments about the centre, O , of the square, if the side of the square be 4 feet.

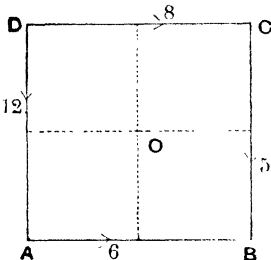
The forces along DA and AB tend to turn the square about O in the positive direction whilst the forces along the sides DC and CB tend to turn it in the negative direction.

The perpendicular distance of O from each force is 2 feet.

Hence the moments of the forces are respectively

$$+6 \times 2, -5 \times 2, -8 \times 2, \text{ and } +12 \times 2.$$

Their algebraic sum is therefore $2[6 - 5 - 8 + 12]$ or 10 units of moment, i.e. 10 times the moment of a force equal to 1 lb. wt. acting at the distance of 1 foot from O .



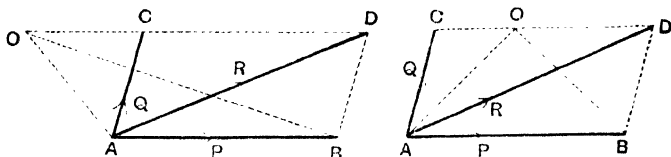
56. Theorem. The algebraic sum of the moments of any two forces about any point in their plane is equal to the moment of their resultant about the same point.

Case I. Let the forces meet in a point.

Let P and Q acting at the point A be the two forces and O the point about which the moments are taken. Draw OC

parallel to the direction of P to meet the line of action of Q in the point C .

Let AC represent Q in magnitude and on the same scale let AB represent P ; complete the parallelogram $ABDC$, and join OA and OB . Then, by the Parallelogram of Forces, AD represents the resultant, R , of P and Q .



(a) If O be without the angle DAC , as in the first figure, we have to shew that

$$2\triangle OAB + 2\triangle OAC = 2\triangle OAD.$$

Since AB and OD are parallel, we have

$$\triangle OAB = \triangle DAB = \triangle ACD.$$

$$\therefore 2\triangle OAB + 2\triangle OAC = 2\triangle ACD + 2\triangle OAC = 2\triangle OAD.$$

(β) If O be within the angle CAD , as in the second figure, we have to shew that

$$2\triangle AOB - 2\triangle AOC = 2\triangle AOD.$$

As in (α), we have

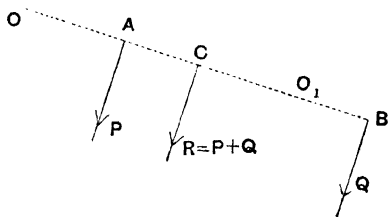
$$\triangle AOB = \triangle DAB = \triangle ACD.$$

$$\therefore 2\triangle AOB - 2\triangle AOC = 2\triangle ACD - 2\triangle AOC = 2\triangle AOD.$$

Case II. Let the forces be parallel.

Let P and Q be two parallel forces and $R (= P + Q)$ their resultant.

From any point O in their plane draw $OACB$ perpendicular to the forces to meet them in A , C , and B respectively.



By Art. 47 we have $P \cdot AC = Q \cdot CB$ (1).

$\therefore$ the sum of the moments of P and Q about O

$$\begin{aligned}
 &= Q \cdot OB + P \cdot OA \\
 &= Q(OC + CB) + P(OC - AC) \\
 &= (P + Q) OC + Q \cdot CB - P \cdot AC \\
 &= (P + Q) \cdot OC, \text{ by equation (1),} \\
 &= \text{moment of the resultant about } O.
 \end{aligned}$$

If the point about which the moments are taken be between the forces, as O_1 , the moments of P and Q have opposite signs.

In this case we have

Algebraic sum of moments of P and Q about O_1

$$\begin{aligned}
 &= P \cdot O_1A - Q \cdot O_1B \\
 &= P(O_1C + CA) - Q(CB - O_1C) \\
 &= (P + Q) \cdot O_1C + P \cdot CA - Q \cdot CB \\
 &= (P + Q) \cdot O_1C, \text{ by equation (1).}
 \end{aligned}$$

The case when the point has any other position, as also the case when the forces have opposite parallel directions, are left for the student to prove for himself.

57. Case 1 of the preceding proposition may be otherwise proved in the following manner:

Let the two forces, P and Q , be represented by AB and AC respectively, and let AD represent the resultant R so that $ABDC$ is a parallelogram.

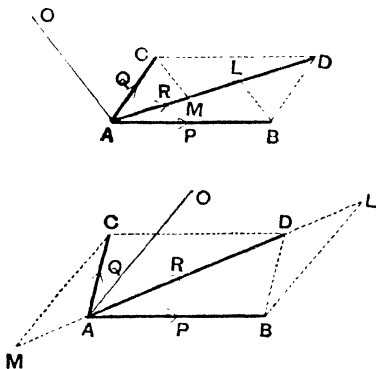
Let O be any point in the plane of the forces. Join OA and draw BL and CM , parallel to OA , to meet AD in L and M respectively.

Since the sides of the triangle ACM are respectively parallel to the sides of the triangle DBL , and since AC is equal to BD ,

$$\therefore AM = LD,$$

$$\therefore \triangle OAM = \triangle OLD.$$

First, let O fall without the angle CAD , as in the first figure.



Then

$$\begin{aligned} & 2\Delta OAB + 2\Delta OAC \\ &= 2\Delta OAL + 2\Delta OAM \\ &= 2\Delta OAL + 2\Delta OLD \\ &= 2\Delta OAD. \end{aligned}$$

Hence the sum of the moments of P and Q is equal to that of R .

Secondly, let O fall within the angle CAD , as in the second figure. The algebraic sum of the moments of P and Q about O

$$\begin{aligned} &= 2\Delta OAB - 2\Delta OAC \\ &= 2\Delta OAL - 2\Delta OAM \\ &= 2\Delta OAL - 2\Delta OLD \\ &= 2\Delta OAD \\ &= \text{moment of } R \text{ about } O. \end{aligned}$$

58. If the point O , about which the moments are taken, lies on the resultant, the moment of the resultant about the point vanishes. In this case the algebraic sum of the moments of the component forces about the given point vanishes, *i.e.* *The moments of two forces about any point on the line of action of their resultant are equal and of opposite sign.*

The student will easily be able to prove this theorem independently from a figure; for, in Art. 56, the point O will be found to coincide with the point D and we have only to shew that the triangles ACO and ABO are now equal, and this is obviously true.

59. Generalised theorem of moments. *If any number of forces in one plane acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant.*

For let the forces be $P, Q, R, S \dots$ and let O be the point about which the moments are taken.

Let P_1 be the resultant of P and Q ,

P_2 be the resultant of P_1 and R ,

P_3 be the resultant of P_2 and S ,

and so on till the resultant is obtained.

Then the moment of P_1 about O = sum of the moments of P and Q (Art. 56);

Also the moment of P_2 about O = sum of the moments of P_1 and R

$$= \text{sum of the moments of } P, Q, \text{ and } R.$$

So the moment of P_3 about O

= sum of the moments of P_2 and S

= sum of the moments of P, Q, R , and S ,

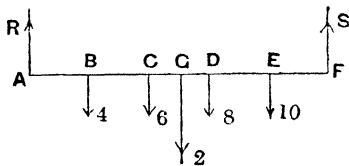
and so on until all the forces have been taken.

Hence the moment of the final resultant

= algebraic sum of the moments of the component forces.

Cor. It follows, similarly as in Art. 58, that the algebraic sum of the moments of any number of forces about a point on the line of action of their resultant is zero; so, conversely, if the algebraic sum of the moments of any number of forces about any point in their plane vanishes, then, either their resultant is zero (in which case the forces are in equilibrium), or the resultant passes through the point about which the moments are taken.

60. Ex. A rod, 5 feet long, supported by two vertical strings attached to its ends has weights of 4, 6, 8, and 10 lbs. hung from it at distances of 1, 2, 3, and 4 feet from one end. If the weight of the rod be 2 lbs., what are the tensions of the strings?



Let AF be the rod, B, C, D , and E the points at which the weights are hung; let G be the middle point; we shall assume that the weight of the rod acts here.

Let R and S be the tensions of the strings. Since the resultant of the forces is zero, its moment about A must be zero.

Hence, by Art. 59, the algebraic sum of the moments about A must vanish.

Therefore $4 \times 1 + 6 \times 2 + 2 \times 2\frac{1}{2} + 8 \times 3 + 10 \times 4 - S \times 5 = 0$,

$$\therefore 5S = 4 + 12 + 5 + 24 + 40 = 85,$$

$$\therefore S = 17.$$

Similarly, taking moments about F , we have

$$5R = 10 \times 1 + 8 \times 2 + 2 \times 2\frac{1}{2} + 6 \times 3 + 4 \times 4 = 65,$$

$$\therefore R = 13.$$

The reaction R may be otherwise obtained. For the resultant of the weights is a weight equal to 30 lbs. and that of R and S is a force equal to $R + S$. But these resultants balance one another.

$$\therefore R + S = 30;$$

$$\therefore R = 30 - S = 30 - 17 = 13.$$

EXAMPLES. VII

1. The side of a square $ABCD$ is 4 feet; along the lines CA , BA , DA , and DB , respectively act forces equal to 4, 3, 2, and 5 lbs. weight; find to the nearest decimal of a foot-pound the algebraic sum of the moments of the forces about C .

2. A pole of 20 feet length is placed with its end on a horizontal plane and is pulled by a string, attached to its upper end and inclined at 30° to the horizon, whose tension is equal to 30 lbs. wt.; find the horizontal force which applied at a point 4 feet above the ground will keep the pole in a vertical position.

3. A uniform iron rod is of length 6 feet and mass 9 lbs., and from its extremities are suspended masses of 6 and 12 lbs. respectively; from what point must the rod be suspended so that it may remain in a horizontal position?

4. A uniform beam is of length 12 feet and weight 50 lbs., and from its ends are suspended bodies of weights 20 and 30 lbs. respectively; at what point must the beam be supported so that it may remain in equilibrium?

5. Masses of 1 lb., 2 lbs., 3 lbs., and 4 lbs. are suspended from a uniform rod, of length 5 ft., at distances of 1 ft., 2 ft., 3 ft., and 4 ft. respectively from one end. If the mass of the rod be 4 lbs., find the position of the point about which it will balance.

6. A uniform rod, 4 ft. in length and weighing 2 lbs., turns freely about a point distant one foot from one end and from that end a weight of 10 lbs. is suspended. What weight must be placed at the other end to produce equilibrium?

7. A heavy uniform beam, 10 feet long, whose mass is 10 lbs., is supported at a point 4 feet from one end; at this end a mass of 6 lbs. is placed; find the mass which, placed at the other end, would give equilibrium.

8. The horizontal roadway of a bridge is 30 feet long, weighs 6 tons, and rests on similar supports at its ends. What is the pressure borne by each support when a carriage, of weight 2 tons, is (1) halfway across, (2) two-thirds of the way across?

9. A light rod, AB , 20 inches long, rests on two pegs whose distance apart is 10 inches. How must it be placed so that the pressures on the pegs may be equal when weights of $2W$ and $3W$ respectively are suspended from A and B ?

10. A light rod, of length 3 feet, has equal weights attached to it, one at 9 inches from one end and the other at 15 inches from the other end; if it be supported by two vertical strings attached to its ends and if the strings cannot support a tension greater than the weight of 1 cwt., what is the greatest magnitude of the equal weights?

11. A heavy uniform beam, whose mass is 40 lbs., is suspended in a horizontal position by two vertical strings each of which can sustain

a tension of 35 lbs. weight. How far from the centre of the beam must a body, of mass 20 lbs., be placed so that one of the strings may just break?

12. A rod, 16 inches long, rests on two pegs, 9 inches apart, with its centre midway between them. The greatest masses that can be suspended in succession from the two ends without disturbing the equilibrium are 4 lbs. and 5 lbs. respectively. Find the weight of the rod and the position of the point at which its weight acts.

13. A straight rod, 2 feet long, is movable about a hinge at one end and is kept in a horizontal position by a thin vertical string attached to the rod at a distance of 8 inches from the hinge and fastened to a fixed point above the rod; if the string can just support a mass of 9 ozs. without breaking, find the greatest mass that can be suspended from the other end of the rod, neglecting the weight of the rod.

14. A tricycle, weighing 5 stone 4 lbs., has a small wheel symmetrically placed 3 feet behind two large wheels which are 3 feet apart; if the centre of gravity of the machine be at a horizontal distance of 9 inches behind the front wheels and that of the rider, whose weight is 9 stone, be 3 inches behind the front wheels, find the pressures on the ground of the different wheels.

15. A front-steering tricycle, of weight 6 stone, has a small wheel symmetrically placed 3 ft. 6 ins. in front of the line joining the two large wheels which are 3 feet apart; if the centre of gravity of the machine be distant horizontally 1 foot in front of the hind wheels and that of the rider, whose weight is 11 stone, be 6 inches in front of the hind wheels, find how the weight is distributed on the different wheels.

16. A dog-cart, loaded with 4 cwt., exerts a pressure on the horse's back equal to 10 lbs. wt.; find the position of the centre of gravity of the load if the distance between the pad and the axle be 6 feet.

17. The wire passing round a telegraph pole is horizontal and the two portions attached to the pole are inclined at an angle of 60° to one another. The pole is supported by a wire attached to the middle point of the pole and inclined at 60° to the horizon; shew that the tension of this wire is $4\sqrt{3}$ times that of the telegraph wire.

18. A cyclist, whose weight is 150 lbs., puts all his weight upon one pedal of his bicycle when the crank is horizontal and the bicycle is prevented from moving forwards. If the length of the crank is 6 inches and the radius of the chain wheel is 4 inches, shew that the tension of the chain is 225 lbs. wt.

19. The horizontal distance between the front and back axles of a motor van is 8 feet, and the force on either axle is not to exceed 2 tons weight. If, when the van is loaded, the centre of gravity is 3 feet in front of the back axle, what is the greatest load possible and what is the thrust on each axle then?

**CHAPTER VI

COUPLES

61. Def. Two equal unlike parallel forces, whose lines of action are not the same, form a couple.

The Arm of a couple is the perpendicular distance between the lines of action of the two forces which form the couple, *i.e.* is the perpendicular drawn from any point lying on the line of action of one of the forces upon the line of action of the other. Thus the arm of the couple (P, P) is the length AB .

The Moment of a couple is the product of one of the forces forming the couple and the arm of the couple.

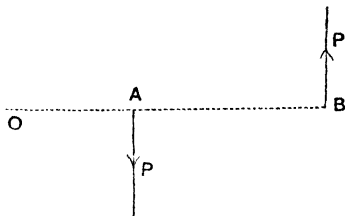
In the figure the moment of the couple is $P \times AB$.

Examples of a couple are the forces applied to the handle of a screw-press, or to the key of a clock in winding it up, or to the handle of a door in opening it.

62. Theorem. *The algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant, and equal to the moment of the couple.*

Let the couple consist of two forces, each equal to P , and let O be any point in their plane.

Draw OAB perpendicular to the lines of action of the forces to meet them in A and B respectively.



The algebraic sum of the moments of the forces about O

$$= P \cdot OB - P \cdot OA = P(OB - OA) = P \cdot AB$$

$$= \text{the moment of the couple, and is therefore the same}$$

whatever be the point O about which the moments are taken.

63. Theorem. *Two couples, acting in one plane upon a rigid body, whose moments are equal and opposite, balance one another.*

Let one couple consist of two forces (P, P) , acting at the ends of an arm p , and let the other couple consist of two forces (Q, Q) , acting at the ends of an arm q .

Case I. Let one of the forces P meet one of the forces Q in a point O , and let the other two forces meet in O' . From O' draw perpendiculars, $O'M$ and $O'N$, upon the forces which do not pass through O' , so that the lengths of these perpendiculars are p and q respectively.

Since the moments of the couples are equal in magnitude, we have

$$P \cdot p = Q \cdot q, \text{ i.e. } P \cdot O'M = Q \cdot O'N.$$

Hence, Art. 58, O' is on the line of action of the resultant of P and Q acting at O , so that OO' is the direction of this resultant.

Similarly, the resultant of P and Q at O' is in the direction $O'O$.

Also these resultants are equal in magnitude; for the forces at O are respectively equal to, and act at the same angle as, the forces at O' .

Hence these two resultants destroy one another, and therefore the four forces composing the two couples are in equilibrium.

Case II. Let the forces composing the couples be all parallel, and let any straight line perpendicular to their directions meet them in the points A, B, C and D , as in the figure, so that we have

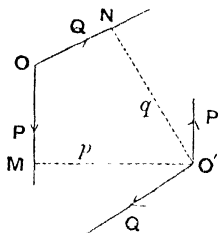
$$P \cdot AB = Q \cdot CD \dots\dots\dots(i).$$

Let L be the point of application of the resultant of Q at C and P at B , so that

$$P \cdot BL = Q \cdot CL \dots\dots\dots(ii).$$

By subtracting (ii) from (i), we have

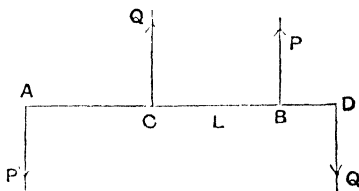
$$P \cdot AL = Q \cdot LD,$$



so that L is the point of application of the resultant of P at A , and Q at D .

But the magnitude of each of these resultants is $(P + Q)$, and they have opposite directions; hence they are in equilibrium.

Therefore the four forces composing the two couples balance.



64. Since two couples in the same plane, of equal but opposite moments, balance, it follows, by reversing the directions of the forces composing one of the couples, that

Any two couples of equal moment in the same plane are equivalent.

It follows also that two like couples of equal moment are equivalent to a couple of double the moment.

65. Theorem. *Any number of couples in the same plane acting on a rigid body are equivalent to a single couple, whose moment is equal to the algebraic sum of the moments of the couples.*

For let the couples consist of forces (P, P) whose arm is p , (Q, Q) whose arm is q , (R, R) whose arm is r , etc. Replace the couple (Q, Q) by a couple whose components have the same lines of action as the forces (P, P) . The magnitude of each of the forces of this latter couple will be X , where $X \cdot p = Q \cdot q$ (Art. 64),

so that
$$X = Q \frac{q}{p}.$$

So let the couple (R, R) be replaced by a couple $(R \frac{r}{p}, R \frac{r}{p})$, whose forces act in the same lines as the forces (P, P) .

Similarly for the other couples.

Hence all the couples are equivalent to a couple, each of whose forces is $P + Q \frac{q}{p} + R \frac{r}{p} + \dots$ acting at an arm p .

The moment of this couple is

$$\left(P + Q \frac{q}{p} + R \frac{r}{p} + \dots \right) \cdot p,$$

$$\text{i.e., } P \cdot p + Q \cdot q + R \cdot r + \dots$$

Hence the original couples are equivalent to a single couple, whose moment is equal to the sum of their moments.

If all the component couples have not the same sign we must give to each moment its proper sign, and the same proof will apply.

Ex. $ABCD$ is a square; along AB and CD act forces of 3 lbs. wt., and along AD and CB forces of 4 lbs. wt., whilst at A and C are applied forces, parallel respectively to BD and DB , each equal to $5\sqrt{2}$ lbs. wt. Find the moment of the couple to which these are equivalent, if the side of the square be 2 feet.

By Art. 54 the moment of the first couple is positive and those of the other two are negative.

The distance $AC = \sqrt{2^2 + 2^2} = 2\sqrt{2}$.

Hence the required moment, by the last article,

$$= 3 \times 2 - 4 \times 2 - 5\sqrt{2} \times AC$$

$$= 6 - 8 - 20 = -22.$$

Hence the equivalent couple is one whose moment is negative and equal to 22 ft.-lbs. wt.

EXAMPLES. VIII

1. $ABCD$ is a square whose side is 2 feet; along AB , BC , CD , and DA act forces equal to 1, 2, 8, and 5 lbs. wt., and along AC and DB forces equal to $5\sqrt{2}$ and $2\sqrt{2}$ lbs. wt.; shew that they are equivalent to a couple whose moment is equal to 16 foot-pounds weight.

2. Along the sides AB and CD of a square $ABCD$ act forces each equal to 2 lbs. weight, whilst along the sides AD and CB act forces each equal to 5 lbs. weight; if the side of the square be 3 feet, find the moment of the couple that will give equilibrium.

3. $ABCDEF$ is a regular hexagon; along the sides AB , CB , DE , and FE act forces respectively equal to 5, 11, 5, and 11 lbs. weight, and along CD and FA act forces, each equal to x lbs. weight. Find x , if the forces be in equilibrium.

4. A horizontal bar AB , without weight, is acted upon by a vertical downward force of 1 lb. weight at A , a vertical upward force of 1 lb. weight at B , and a downward force of 5 lbs. weight at a given point C inclined to the bar at an angle of 30° . Find at what point of the bar a force must be applied to balance these, and find also its magnitude and direction.

CHAPTER VII

EQUILIBRIUM OF A RIGID BODY ACTED ON BY THREE FORCES IN A PLANE

66. In the present chapter we shall discuss some simple cases of the equilibrium of a rigid body acted upon by three forces lying in a plane.

By the help of the theorem of the next article we shall find that the conditions of equilibrium reduce to those of a single particle.

67. Theorem. *If three forces, acting in one plane upon a rigid body, keep it in equilibrium, they must either meet in a point or be parallel.*

If the forces be not all parallel, at least two of them must meet; let these two be P and Q , and let their directions meet in O .

The third force R will then pass through the point O .

Since the algebraic sum of the moments of any number of forces about a point in their plane is equal to the moment of their resultant (Art. 59),

therefore the sum of the moments of P , Q , and R about O is equal to the moment of their resultant.

But this resultant vanishes since the forces are in equilibrium.

Hence the sum of the moments of P , Q , and R about O is zero.

But, since P and Q both pass through O , their moments about O vanish.

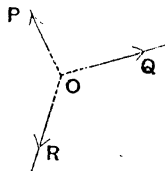
Hence the moment of R about O vanishes.

Hence by Art. 58, since R is not zero, its line of action must pass through O .

Hence the forces meet in a point.

Otherwise. The resultant of P and Q must be some force passing through O .

But, since the forces P , Q , and R are in equilibrium, this resultant must balance R .



But two forces cannot balance unless they have the same line of action.

Hence the line of action of R must pass through O .

68. By the preceding theorem we see that the conditions of equilibrium of three forces, acting in one plane, are easily obtained. For the three forces must meet in a point; and by using Lami's Theorem (Art. 33), or by resolving the forces in two directions at right angles (Art. 37), we can obtain the required conditions.

Ex. 1. A heavy uniform rod AB is hinged at A to a fixed point, and rests in a position inclined at 60° to the horizontal, being acted upon by a horizontal force F applied at the lower end B ; find the action at the hinge and the magnitude of F .

Let the vertical through C , the middle point of the rod, meet the horizontal line through B in the point D and let the weight of the rod be W .

There are only three forces acting on the rod, viz., the force F , the weight W , and the unknown reaction, P , of the hinge.

These three forces must therefore meet in a point.

Now F and W meet at D ; hence the direction of the action at the hinge must be the line DA .

Draw AE perpendicular to EB .

Let $AC = CB = a$.

$$\therefore BE = AB \cos 60^\circ = 2a \times \frac{1}{2} = a.$$

Hence $AE = \sqrt{AB^2 - BE^2} = a\sqrt{3},$

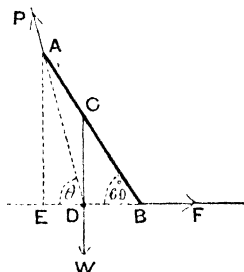
and $AD = \sqrt{AE^2 + DE^2} = \sqrt{3a^2 + \frac{a^2}{4}} = \frac{a}{2}\sqrt{13}.$

Since the triangle ADE has its sides respectively parallel to the forces F , P , and W , we have

$$\frac{P}{DA} = \frac{F}{ED} = \frac{W}{AE}.$$

$$\therefore \frac{P}{\frac{a}{2}\sqrt{13}} = \frac{F}{\frac{a}{2}} = \frac{W}{a\sqrt{3}}.$$

Ex. 2. A uniform rod, AB , is inclined at an angle of 60° to the vertical with one end A resting against a smooth vertical wall, being supported by a string attached to a point C of the rod, distant 1 foot from B , and also to a ring in the wall vertically above A ; if the length



of the rod be 4 feet, find the position of the ring and the inclination and tension of the string.

Let the perpendicular to the wall through A and the vertical line through the middle point, G , of the rod meet in O .

The third force, the tension T of the string, must therefore pass through O . Hence CO produced must pass through D , the position of the ring.

Let the angle CDA be θ , and draw CE horizontal to meet OG in E and the wall in F .

$$\begin{aligned}\text{Then } \tan \theta &= \tan COE = \frac{CE}{OE} = \frac{CG \sin CGE}{AF} \\ &= \frac{1 \cdot \sin 60^\circ}{3 \cdot \cos 60^\circ} = \frac{1}{\sqrt{3}}. \\ \therefore \theta &= 30^\circ.\end{aligned}$$

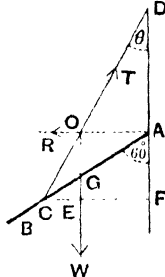
$$\therefore ACD = 60^\circ - \theta = 30^\circ.$$

Hence $AD = AC = 3$ feet, giving the position of the ring.

If R be the reaction of the wall, and W be the weight of the beam, we have, since the forces are proportional to the sides of the triangle AOD ,

$$\begin{aligned}\frac{T}{OD} &= \frac{R}{AO} = \frac{W}{DA}. \\ \therefore T &= W \frac{OD}{DA} = \frac{W}{\cos 30^\circ} = W \cdot \frac{2}{\sqrt{3}},\end{aligned}$$

$$\text{and } R = W \frac{AO}{DA} = W \tan 30^\circ = W \cdot \frac{1}{\sqrt{3}}.$$



EXAMPLES. IX

1. A uniform rod, AB , of weight W , is movable in a vertical plane about a hinge at A , and is sustained in equilibrium by a weight P attached to a string BCP passing over a smooth peg C , AC being vertical; if AC be equal to AB , shew that $P = W \cos ACB$, and that the action at the hinge is $W \sin ACB$.

2. A uniform rod can turn freely about one of its ends, and is pulled aside from the vertical by a horizontal force acting at the other end of the rod and equal to half its weight; at what inclination to the vertical will the rod rest?

3. A rod AB , hinged at A , is supported in a horizontal position by a string BC , making an angle of 45° with the rod, and the rod has a mass of 10 lbs. suspended from B . Neglecting the weight of the rod, find the tension of the string and the action at the hinge.

4. A uniform heavy rod AB has the end A in contact with a smooth vertical wall, and one end of a string is fastened to the rod at a point C , such that $AC = \frac{1}{4}AB$, and the other end of the string is fastened to the wall; find the length of the string, if the rod rest in a position inclined at an angle to the vertical.

THREE FORCES ACTING ON A BODY 51

5. ACB is a uniform rod, of weight W ; it is supported (B being uppermost) with its end A against a smooth vertical wall AD by means of a string CD , DB being horizontal and CD inclined to the wall at an angle of 30° . Find the tension of the string, and the reaction of the wall, and prove that $AC = \frac{1}{3}AB$.

6. A uniform rod, AB , resting with one end A against a smooth vertical wall is supported by a string BC which is tied to a point C vertically above A and to the other end B of the rod. Draw a diagram shewing the lines of action of the forces which keep the rod in equilibrium, and shew that the tension of the string is greater than the weight of the rod.

7. A ladder, 14 feet long and weighing 50 lbs., rests with one end against the foot of a vertical wall and from a point 4 feet from the upper end a cord which is horizontal runs to a point 6 feet above the foot of the wall. Find the tension of the cord and the reaction at the lower end of the ladder.

8. A smooth hemispherical bowl, of diameter a , is placed so that its edge touches a smooth vertical wall; a heavy rod is in equilibrium, inclined at 60° to the horizon, with one end resting on the inner surface of the bowl, and the other end resting against the wall; shew that the length of the rod must be $a + \frac{a}{\sqrt{13}}$.

9. A sphere, of given weight W , rests between two smooth planes, one vertical and the other inclined at a given angle α to the vertical; find the reactions of the planes.

10. A solid sphere rests upon two parallel bars which are in the same horizontal plane, the distance between the bars being equal to the radius of the sphere; find the reaction of each bar.

11. A smooth sphere is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being attached to a point in the wall; if the length of the string be equal to the radius of the sphere, find the inclination of the string to the vertical, the tension of the string, and the reaction of the wall.

12. A picture of given weight, hanging vertically against a smooth wall, is supported by a string passing over a smooth peg driven into the wall; the ends of the string are fastened to two points in the upper rim of the frame which are equidistant from the centre of the rim, and the angle at the peg is 60° ; compare the tension in this case with what it will be when the string is shortened to two-thirds of its length.

13. A picture, of 40 lbs. wt., is hung, with its upper and lower edges horizontal, by a cord fastened to the two upper corners and passing over a nail, so that the parts of the cord at the two sides of the nail are inclined to one another at an angle of 60° . Find the tension of the cord in lbs. weight.

14. A picture hangs symmetrically by means of a string passing over a nail and attached to two rings in the picture; what is the tension of the string when the picture weighs 10 lbs., if the string be 4 feet long and the nail distant 1 ft. 6 inches from the horizontal line joining the rings?

15. Two equal heavy spheres, of 1 inch radius, are in equilibrium within a smooth spherical cup of 3 inches radius. Shew that the action between the cup and one sphere is double that between the two spheres.

The following examples are to be solved graphically.

16. A heavy beam, AB , 10 feet long, is supported, A uppermost, by two ropes attached to it at A and B which are respectively inclined at 55° and 50° to the horizontal; if AB be inclined at 20° to the horizontal, find at what distance from A its centre of gravity is. Also, if its weight be 200 lbs., find the tensions of the two ropes.

17. A light rod AB , of length 2 feet, is smoothly jointed to a fixed support at A and rests horizontally; at D , where $AD=9$ inches, it carries a weight of 10 lbs., being supported by a light rod CB , where C is exactly underneath A and $AC=6$ inches; find the thrust in the rod CB .

18. AB is a uniform beam turning on a pivot at C and kept in equilibrium by a light string AD attached to the highest point A and to a point D vertically below C . If $AB=3$ ft., $AC=1$ ft., $CD=2$ ft., and $DA=2.7$ ft., and the weight of the beam be 10 lbs., find the tension of the string and the reaction of the pivot.

19. A cantilever consists of a horizontal rod AB hinged to a fixed support at A , and a rod DC hinged at a point C of AB and also hinged to a fixed point D vertically below A . A weight of 1 cwt. is attached at B ; find the actions at A and C , given that $AB=6$ ft., $AC=2$ ft., and $AD=3$ ft., the weights of the rods being neglected.

20. The plane of a kite is inclined at 50° to the horizon, and its weight is 10 lbs. The resultant thrust of the air on the kite is perpendicular to it, and acts at a point 8 inches above its centre of gravity, and the string is tied at a point 10 inches above the centre of gravity. Find the tension of the string and the thrust of the air.

CHAPTER VIII

CENTRE OF GRAVITY

69. EVERY particle of matter is attracted to the centre of the Earth, and the force with which the Earth attracts any particle to itself is, as we shall see in Dynamics, proportional to the mass of the particle.

Any body may be considered as an agglomeration of particles.

If the body be small, compared with the Earth, the lines joining its component particles to the centre of the Earth will be very approximately parallel, and, within the limits of this book, we shall consider them to be absolutely parallel.

On every particle, therefore, of a rigid body there is acting a force vertically downwards which we call its weight.

These forces may by the process of compounding parallel forces, Art. 49, be compounded into a single force, equal to the sum of the weights of the particles, acting at some definite point of the body. Such a point is called the centre of gravity of the body.

Centre of gravity. Def. *The centre of gravity of a body, or system of particles rigidly connected together, is that point through which the line of action of the weight of the body always passes, in whatever position the body is placed.*

70. Every body, or system of particles rigidly connected together, has a centre of gravity.

Let $A, B, C, D \dots$ be a system of particles whose weights are $w_1, w_2, w_3 \dots$

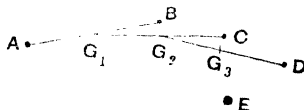
Join AB , and divide it at G_1 so that

$$AG_1 : G_1B :: w_2 : w_1.$$

Then parallel forces w_1 and w_2 , acting at A and B , are, by Art. 47, equivalent to a force $(w_1 + w_2)$ acting at G_1 .

Join G_1C , and divide it at G_2 so that

$$G_1G_2 : G_2C :: w_3 : w_1 + w_2.$$



Then parallel forces, $(w_1 + w_2)$ at G_1 and w_3 at C , are equivalent to a force $(w_1 + w_2 + w_3)$ acting at G_2 .

Hence the forces w_1 , w_2 , and w_3 may be supposed to be applied at G_2 without altering their effect.

Similarly, dividing G_2D in G_3 so that

$$G_2G_3 : G_3D :: w_4 : w_1 + w_2 + w_3,$$

we see that the resultant of the four weights at A , B , C , and D is equivalent to a vertical force $(w_1 + w_2 + w_3 + w_4)$ acting at G_3 .

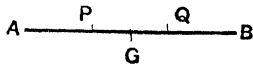
Proceeding in this way, we see that the weights of any number of particles composing any body may be supposed to be applied at some point of the body without altering their effect.

*** * 71.** Since the construction for the position of the resultant of parallel forces depends only on the point of application and magnitude, and not on the direction of the forces, the point we finally arrive at is the same if the body be turned through any angle; for the weights of the portions of the body are still parallel, although they have not the same direction, relative to the body, in the two positions.

We can hence shew that a body can only have one centre of gravity. For, if possible, let it have two centres of gravity G and G_1 . Let the body be turned, if necessary, until GG_1 be horizontal. We shall then have the resultant of a system of vertical forces acting both through G and through G_1 . But the resultant force, being itself necessarily vertical, cannot act in the horizontal line GG_1 .

Hence there can be only one centre of gravity.

72. Centre of gravity of a uniform rod. Let AB be a uniform rod, and G its middle point.



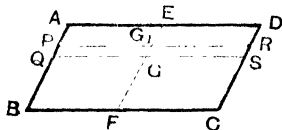
Take any point P of the rod between G and A , and a point Q in GB , such that

$$GQ = GP.$$

The centre of gravity of equal particles at P and Q is clearly G ; also, for every particle between G and A , there is an equal particle at an equal distance from G , lying between G and B .

The centre of gravity of each of these pairs of particles is at G ; therefore the centre of gravity of the whole rod is at G .

73. Centre of gravity of a uniform parallelogram. Let $ABCD$ be a parallelogram, and let E and F be the middle points of AD and BC .



Divide the parallelogram into a very large number of strips, by means of lines parallel to AD , of which PR and QS are any consecutive pair. Then $PQSR$ may be considered to be a uniform straight line, whose centre of gravity is at its middle point G_1 .

So the centre of gravity of all the other strips lies on EF , and hence the centre of gravity of the whole figure lies on EF .

So, by dividing the parallelogram by lines parallel to AB , we see that the centre of gravity lies on the line joining the middle points of the sides AB and CD .

Hence the centre of gravity is at G the point of intersection of these two lines.

G is clearly also the point of intersection of the diagonals of the parallelogram.

74. It is clear from the method of the two previous articles that, if in a uniform body we can find a point G such that the body can be divided into pairs of particles balancing about it, then G must be the centre of gravity of the body.

The centre of gravity of a uniform circle, or uniform sphere, is therefore its centre.

It is also clear that if we can divide a lamina into strips, the centres of gravity of which all lie on a straight line, then the centre of gravity of the lamina must lie on that line.

Similarly, if a body can be divided into portions, the centres of gravity of which lie in a plane, the centre of gravity of the whole must lie in that plane.

75. Centre of gravity of a uniform triangular lamina. Let ABC be the triangular lamina, and let D and E be the middle points of the sides BC and CA . Join AD and BE , and let them meet in G . Then G will be the centre of gravity of the triangle.

Let B_1C_1 be *any* line parallel to the base BC meeting AD in D_1 .

As in the case of the parallelogram, the triangle may be considered to be made up of a very large number of strips, such as B_1C_1 , all parallel to the base BC .

Since B_1C_1 and BC are parallel, the triangles AB_1D_1 and ABD are equiangular; so also the triangles AD_1C_1 and ADC are equiangular.

Hence
$$\frac{B_1D_1}{BD} = \frac{AD_1}{AD} = \frac{D_1C_1}{DC}. \quad (\text{App. I, Art. 3.})$$

But $BD = DC$; therefore $B_1D_1 = D_1C_1$. Hence the centre of gravity of the strip B_1C_1 lies on AD .

So the centres of gravity of all the other strips lie on AD , and hence the centre of gravity of the triangle lies on AD .

Join BE , and let it meet AD in G .

By dividing the triangle into strips parallel to AC we see, similarly, that the centre of gravity lies on BE .

Hence the required centre of gravity must be at G .

Since D is the middle point of BC and E is the middle point of CA , therefore DE is parallel to AB .

Hence the triangles GDE and GAB are equiangular,

$$\therefore \frac{GD}{GA} = \frac{DE}{AB} = \frac{CE}{CA} = \frac{1}{2},$$

so that $2GD = GA$, and $3GD = GA + GD = AD$.

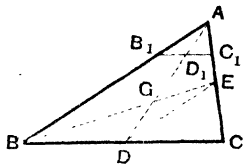
$$\therefore GD = \frac{1}{3}AD.$$

Hence the centre of gravity of a triangle is on the line joining the middle point of any side to the opposite vertex at one-third the distance of the vertex from that side.

***76.** *The centre of gravity of any triangular lamina is the same as that of three equal particles placed at the vertices of the triangle.*

Taking the figure of Art. 75, the centre of gravity of two equal particles, each equal to w , at B and C , is at D the middle point of BC ; also the centre of gravity of $2w$ at D and w at A divides the line DA in the ratio of $1:2$ (Art. 47). But G , the centre of gravity of the lamina, divides DA in the ratio of $1:2$.

Hence the centre of gravity of the three particles is the same as that of the lamina.



***77.** The position of the centre of gravity of some other bodies may be stated here.

The centre of gravity of a **pyramid** on any base is on the line joining the vertex to the centre of gravity of the base and divides this line in the ratio of 3:1.

The centre of gravity of a solid **cone** is on its axis at a distance from the base equal to $\frac{1}{4}$ of its altitude; if the cone be hollow, the distance is $\frac{1}{3}$ of the altitude.

The centre of gravity of a solid **hemisphere** of radius r is on that radius which is perpendicular to its plane face at a distance $\frac{3r}{8}$ from the centre. If the hemisphere be hollow, this distance is $\frac{r}{2}$.

EXAMPLES. X

1. An isosceles triangle has its equal sides of length 5 feet and its base of length 6 feet; find the distance of the centre of gravity from each of its angular points.

2. The sides of a triangular lamina are 6, 8, and 10 feet in length; find the distance of the centre of gravity from each of its angular points.

3. D is the middle point of the base BC of a triangle ABC ; shew that the distance between the centres of gravity of the triangles ABD and ACD is $\frac{1}{3}BC$.

4. A heavy triangular plate ABC lies on the ground; if a vertical force applied at the point A be just great enough to begin to lift that vertex from the ground, shew that the same force will suffice, if applied at B or C .

5. The base of a triangle is fixed, and the vertex moves on a given straight line; shew that the centre of gravity also moves on a straight line.

6. A uniform equilateral triangular plate is suspended by a string attached to a point in one of its sides, which divides the side in the ratio 2:1; find the inclination of this side to the vertical.

78. General formulae for the determination of the centre of gravity.

In the following articles will be obtained formulae giving the position of the centre of gravity of any system of particles, whose position and weights are known.

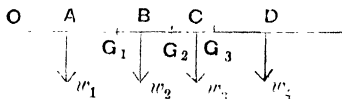
Theorem. *If a system of particles whose weights are $w_1, w_2, \dots w_n$ be on a straight line, and if their distances measured from a fixed point O in the line be*

$$x_1, x_2, \dots x_n,$$

the distance, $\bar{x}$, of their centre of gravity from the fixed point is given by

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \dots + w_nx_n}{w_1 + w_2 + \dots + w_n}$$

Let $A, B, C, D \dots$ be the particles and let the centre of gravity of w_1 and w_2 at A and B be G_1 ; let the centre of gravity of $(w_1 + w_2)$ at G_1 and w_3 at C be G_2 , and so for the other particles of the system.



By Art. 70, we have $w_1 \cdot AG_1 = w_2 \cdot G_1B$.

$$\therefore w_1 (OG_1 - OA) = w_2 (OB - OG_1).$$

Hence $(w_1 + w_2) \cdot OG_1 = w_1 \cdot OA + w_2 \cdot OB,$

$$\text{i.e. } OG_1 = \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2} \dots \dots \dots (1).$$

Similarly, since G_2 is the centre of gravity of $(w_1 + w_2)$ at G_1 and w_3 at C , we have

$$\begin{aligned} OG_2 &= \frac{(w_1 + w_2) \cdot OG_1 + w_3 \cdot OC}{(w_1 + w_2) + w_3} \\ &= \frac{w_1 x_1 + w_2 x_2 + w_3 x_3}{w_1 + w_2 + w_3}, \text{ by (1).} \end{aligned}$$

$$\begin{aligned} \text{So } OG_3 &= \frac{(w_1 + w_2 + w_3) \cdot OG_2 + w_4 \cdot OD}{(w_1 + w_2 + w_3) + w_4} \\ &= \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4}{w_1 + w_2 + w_3 + w_4}. \end{aligned}$$

Proceeding in this manner we easily have

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \dots + w_n x_n}{w_1 + w_2 + \dots + w_n},$$

whatever be the number of the particles in the system.

Otherwise, The above formula may be obtained by the use of Art. 59. For the weights of the particles form a system of parallel forces whose resultant is equal to their sum, viz. $w_1 + w_2 + \dots + w_n$. Also the sum of the moments of these forces about any point in their plane is the same as the moment of their resultant. But the sum of the moments of the forces about the fixed point O is

$$w_1 x_1 + w_2 x_2 + \dots + w_n x_n.$$

Also, if $\bar{x}$ be the distance of the centre of gravity from O , the moment of the resultant is

$$(w_1 + w_2 + \dots + w_n) \times \bar{x}.$$

Hence $\bar{x} (w_1 + w_2 + \dots + w_n) = w_1 x_1 + w_2 x_2 + \dots + w_n x_n;$

$$\text{i.e., } \bar{x} = \frac{w_1 x_1 + w_2 x_2 + \dots + w_n x_n}{w_1 + w_2 + \dots + w_n}.$$

79. Ex. 1. A rod AB , 2 feet in length, and of weight 5 lbs., is trisected in the points C and D , and at the points A , C , D , and B are placed particles of 1, 2, 3, and 4 lbs. weight respectively; find what point of the rod must be supported so that the rod may rest in any position, i.e., find the centre of gravity of the system.

Let G be the middle point of the rod, and let the fixed point O of the previous article be taken to coincide with the end A of the rod. The quantities x_1, x_2, x_3, x_4 , and x_5 are in this case 0, 8, 12, 16, and 24 inches respectively.

Hence, if X be the point required, we have

$$\begin{aligned} AX &= \frac{1 \cdot 0 + 2 \cdot 8 + 5 \cdot 12 + 3 \cdot 16 + 4 \cdot 24}{1 + 2 + 5 + 3 + 4} \\ &= \frac{220}{15} = 14\frac{2}{3} \text{ inches.} \end{aligned}$$

Ex. 2. If, in the previous question, the body at B be removed and another body be substituted, find the weight of this unknown body so that the new centre of gravity may be at the middle point of the rod.

Let y lbs. be the required weight.

Since the distance of the new centre of gravity from A is to be 12 inches, we have

$$\begin{aligned} 12 &= \frac{1 \cdot 0 + 2 \cdot 8 + 5 \cdot 12 + 3 \cdot 16 + y \cdot 24}{1 + 2 + 5 + 3 + y} = \frac{124 + 24y}{11 + y} \\ \therefore 132 + 12y &= 124 + 24y \\ \therefore y &= \frac{2}{3}. \end{aligned}$$

Ex. 3. To the end of a rod, whose length is 2 feet and whose weight is 3 lbs., is attached a sphere, of radius 2 inches and weight 10 lbs.; find the position of the centre of gravity of the compound body.

Let OA be the rod, G_1 its middle point, G_2 the centre of the sphere, and G the required point.

$$\text{Then} \quad OG = \frac{3 \cdot OG_1 + 10 \cdot OG_2}{3 + 10}.$$

But $OG_1 = 12$ inches and $OG_2 = 26$ inches.

$$\therefore OG = \frac{3 \cdot 12 + 10 \cdot 26}{3 + 10} = \frac{296}{13} = 22\frac{10}{13} \text{ inches.}$$

EXAMPLES. XI

1. A straight rod, 1 foot in length and of mass 1 ounce, has an ounce of lead fastened to it at one end, and another ounce fastened to it at a distance from the other end equal to one-third of its length; find the centre of gravity of the system.

2. A uniform bar, 3 feet in length and of mass 6 ounces, has 3 rings, each of mass 3 ounces, at distances 3, 15, and 21 inches from one end. About what point of the bar will the system balance?

3. A uniform rod AB is 4 feet long and weighs 3 lbs. One lb. is attached at A , 2 lbs. at a point distant 1 foot from A , 3 lbs. at 2 feet from A , 4 lbs. at 3 feet from A , and 5 lbs. at B . Find the distance from A of the centre of gravity of the system.

4. A telescope consists of 3 tubes, each 10 inches in length, one within the other, and of weights 8, 7, and 6 ounces. Find the position of the centre of gravity when the tubes are drawn out at full length.

5. Twelve heavy particles at equal intervals of one inch along a straight rod weigh 1, 2, 3, ..., 12 grains respectively; find their centre of gravity, neglecting the weight of the rod.

6. The four silver coins from one florin downwards are placed in a straight line with equal distances of 6 inches between their centres. Find their centre of gravity.

7. A rod, of uniform thickness, has one-half of its length composed of one metal and the other half composed of a different metal, and the rod balances about a point distant one-third of its whole length from one end; compare the weights of equal quantities of the metals.

8. A cylindrical vessel, one foot in diameter and one foot high, is made of thin sheet metal of uniform thickness. If it be half filled with water where will be the common centre of gravity of the vessel and water, assuming the weight of the vessel to be $\frac{1}{5}$ th of the contained water?

9. A rod, 12 feet long, has a mass of 1 lb. suspended from one end, and, when 15 lbs. is suspended from the other end, it balances about a point distant 3 ft. from that end; if 8 lbs. be suspended there, it balances about a point 4 ft. from that end. Find the weight of the rod and the position of its centre of gravity.

80. Theorem. *If a system of particles, whose weights are $w_1, w_2, \dots, w_n$, lie in a plane, and if OX and OY be two fixed straight lines in the plane at right angles, and if the distances of the particles from OX be $y_1, y_2, \dots, y_n$, and the distance of their centre of gravity be $\bar{y}$, then*

$$\bar{y} = \frac{w_1 y_1 + w_2 y_2 + \dots + w_n y_n}{w_1 + w_2 + \dots + w_n}.$$

Similarly, if the distances of the particles from OY be $x_1, x_2, \dots, x_n$ and that of their centre of gravity be $\bar{x}$, then

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \dots + w_n x_n}{w_1 + w_2 + \dots + w_n}.$$

Let $A, B, C, \dots$ be the particles, and $AL, BM, CN, \dots$ the perpendiculars on OX .

Let G_1 be the centre of gravity of w_1 and w_2 , G_2 the centre of gravity of $(w_1 + w_2)$ at G_1 and w_3 at C , and so on.

Also let G be the final point thus arrived at, *i.e.* the centre of gravity of all the particles.

Since the resultant weight $(w_1 + w_2 + \dots + w_n)$ acting at G is equivalent to the component forces $w_1, w_2, \dots$ the resultant would, if the line OX be supposed to be a fixed axis, have the same moment about this fixed axis that the component weights have.

But the moment of the resultant is

$$(w_1 + w_2 + \dots + w_n) \bar{y},$$

and the sum of the moments of the weights is

$$w_1 y_1 + w_2 y_2 + \dots + w_n y_n.$$

Hence
$$\bar{y} = \frac{w_1 y_1 + w_2 y_2 + \dots + w_n y_n}{w_1 + w_2 + \dots + w_n}.$$

In a similar manner we should have

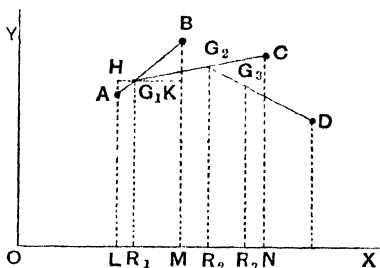
$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \dots + w_n x_n}{w_1 + w_2 + \dots + w_n}.$$

The theorem of this article may be put somewhat differently as follows:

The distance of the centre of gravity from any line in the plane of the particles is equal to a fraction, whose numerator is the sum of the products of each weight into its distance from the given line, and whose denominator is the sum of the weights.

In other words, the distance of the centre of gravity is equal to the **average** distance of the particles.

81. Ex. 1. A square lamina, whose weight is 10 lbs., has attached to its angular points particles whose weights, taken in order, are 3, 6, 5, and 1 lbs. respectively. Find the position of the centre of gravity of the system, if the side of the lamina is 25 inches.



Let the particles be placed at the angular points O , A , B , and C . Let the two fixed lines from which the distances are measured be OA and OC .

The weight of the lamina acts at its centre D . Let G be the required centre of gravity and draw DL and GM perpendicular to OX .

The distances of the points O , A , B , C , and D from OX are clearly 0, 0, 25, 25, and $12\frac{1}{2}$ inches respectively.

$$\begin{aligned}\therefore MG = \bar{y} &= \frac{3 \cdot 0 + 6 \cdot 0 + 5 \cdot 25 + 1 \cdot 25 + 10 \cdot 12\frac{1}{2}}{3 + 6 + 5 + 1 + 10} \\ &= \frac{275}{25} = 11 \text{ ins.}\end{aligned}$$

So the distances of the particles from OY are 0, 25, 25, 0, and $12\frac{1}{2}$ inches respectively.

$$\therefore OM = \bar{x} = \frac{3 \cdot 0 + 6 \cdot 25 + 5 \cdot 25 + 1 \cdot 0 + 10 \cdot 12\frac{1}{2}}{3 + 6 + 5 + 1 + 10} = \frac{400}{25} = 16 \text{ ins.}$$

Hence the required point may be obtained by measuring 16 inches from O along OA and then erecting a perpendicular of length 11 inches.

Ex. 2. OAB is an isosceles weightless triangle, whose base OA is 6 inches and whose sides are each 5 inches; at the points O , A , and B are placed particles of weights 1, 2, and 3 lbs.; find their centre of gravity.

Let the fixed line OX coincide with OA and let OY be a perpendicular to OA through the point O .

If BL be drawn perpendicular to OA , then $OL = 3$ ins., and

$$LB = \sqrt{5^2 - 3^2} = 4 \text{ ins.}$$

Hence, if G be the required centre of gravity and GM be drawn perpendicular to OX , we have

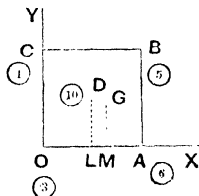
$$OM = \frac{1 \cdot 0 + 2 \cdot 6 + 3 \cdot 3}{1 + 2 + 3} = \frac{21}{6} = 3\frac{1}{2} \text{ inches,}$$

$$\text{and } MG = \frac{1 \cdot 0 + 2 \cdot 0 + 3 \cdot 4}{1 + 2 + 3} = \frac{12}{6} = 2 \text{ inches.}$$

Hence the required point is obtained by measuring a distance $3\frac{1}{2}$ inches from O along OA and then erecting a perpendicular of length 2 inches.

82. Centre of Parallel forces.

The methods and formulae of Arts. 78 and 80 will apply not only to weights, but also to any system of parallel forces and will determine the position of the resultant of any such system. The magnitude of the resultant is the sum of the forces. Each force must, of course, be taken with its proper sign prefixed.



There is one case in which we obtain no satisfactory result; if the algebraic sum of the forces be zero, the resultant force is zero, and the formulae of Art. 80 give

$$\bar{x} = \infty, \text{ and } \bar{y} = \infty.$$

In this case the system of parallel forces is, as in Art. 61, equivalent to a couple.

EXAMPLES. XII

1. Particles of 1, 2, 3, and 4 lbs. weight are placed at the angular points of a square; find the distance of their centre of gravity from the centre of the square.

2. At two opposite corners A and C of a square $ABCD$ weights of 2 lbs. each are placed, and at B and D are placed 1 and 7 lbs. respectively; find their centre of gravity.

3. Particles of 5, 6, 9, and 7 lbs. respectively are placed at the corners A , B , C , and D of a horizontal square, the length of whose side is 27 inches; find where a single force must be applied to preserve equilibrium.

4. Five masses of 1, 2, 3, 4, and 5 ounces respectively are placed on a square table. The distances from one edge of the table are 2, 4, 6, 8, and 10 inches and from the adjacent edge 3, 5, 7, 9, and 11 inches respectively. Find the distance of the centre of gravity from the two edges.

5. Weights proportional to 1, 2, and 3 are placed at the corners of an equilateral triangle, whose side is of length a ; find the distance of their centre of gravity from the first weight.

Find the distance also if the weights be proportional to 11, 13, and 6.

6. ABC is an equilateral triangle of side 2 feet. At A , B , and C are placed weights proportional to 5, 1, and 3, and at the middle points of the sides BC , CA , and AB weights proportional to 2, 4, and 6; shew that their centre of gravity is distant 16 inches from B .

7. Equal masses, each 1 oz., are placed at the angular points of a heavy triangular lamina, and also at the middle points of its sides; find the position of the centre of gravity of the masses.

8. ABC is a triangle right-angled at A , AB being 12 and AC 15 inches; weights proportional to 2, 3, and 4 respectively are placed at A , C , and B ; find the distances of their centre of gravity from B and C .

9. Particles, of masses 4, 1, and 1 lbs., are placed at the angular points of a triangle; shew that the centre of gravity of the particles bisects the distance between the centre of gravity and one of the vertices of the triangle.

10. To the vertices A , B , and C of a uniform triangular plate, whose mass is 3 lbs., and whose centre of gravity is G , particles of

masses 2 lbs., 2 lbs., and 11 lbs., are attached; shew that the centre of gravity of the system is the middle point of GC .

11. Find the centre of parallel forces equal respectively to P , $2P$, $3P$, $4P$, $5P$, and $6P$, the points of application of the forces being at distances 1, 2, 3, 4, 5, and 6 inches respectively from a given point A measured along a given line AB .

12. At the angular points of a square, taken in order, there act parallel forces in the ratios 1:3:5:7; find the distance from the centre of the square of the point at which their resultant acts.

13. A , B , C , and D are the angles of a parallelogram taken in order; like parallel forces proportional to 6, 10, 14, and 10 respectively act at A , B , C , and D ; shew that the centre and resultant of these parallel forces remain the same, if, instead of these forces, parallel forces, proportional to 8, 12, 16, and 4, act at the points of bisection of the sides AB , BC , CD , and DA respectively.

83. Given the centre of gravity of the two portions of a body, to find the centre of gravity of the whole body.

Let the given centres of gravity be G_1 and G_2 , and let the weights of the two portions be W_1 and W_2 ; the required point G , by Art. 70, divides G_1G_2 so that

$$G_1G : GG_2 :: W_2 : W_1.$$

The point G may also be obtained by the use of Art. 78.

Ex. On the same base AB , and on opposite sides of it, isosceles triangles CAB and DAB are described whose altitudes are 12 inches and 6 inches respectively. Find the distance from AB of the centre of gravity of the quadrilateral $CADB$.

Let CLD be the perpendicular to AB , meeting it in L , and let G_1 and G_2 be the centres of gravity of the two triangles CAB and DAB respectively. Hence

$$CG_1 = \frac{2}{3} \cdot CL = 8,$$

and $CG_2 = CL + LG_2 = 12 + 2 = 14.$

The weights of the triangles are proportional to their areas, i.e., to $\frac{1}{2}AB \cdot 12$ and $\frac{1}{2}AB \cdot 6$.

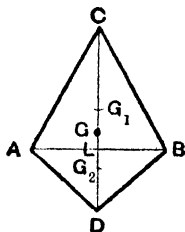
If G be the centre of gravity of the whole figure, we have

$$CG = \frac{\Delta CAB \times CG_1 + \Delta DAB \times CG_2}{\Delta CAB + \Delta DAB} \\ = \frac{\frac{1}{2}AB \cdot 12 \times 8 + \frac{1}{2}AB \cdot 6 \times 14}{\frac{1}{2}AB \cdot 12 + \frac{1}{2}AB \cdot 6} = \frac{48 + 42}{6 + 3} = \frac{90}{9} = 10.$$

Hence

$$LG = CL - CG = 2 \text{ inches.}$$

This result may be verified experimentally by cutting the figure out of thin cardboard.



84. Given the centre of gravity of the whole of a body and of a portion of the body, to find the centre of gravity of the remainder.

Let G be the centre of gravity of a body $ABCD$, and G_1 that of the portion ADC .

Let W be the weight of the whole body and W_1 that of the portion ADC , so that $W_2 (= W - W_1)$ is the weight of the portion ABC .

Let G_2 be the centre of gravity of the portion ABC . Since the two portions of the body make up the whole, therefore W_1 at G_1 and W_2 at G_2 must have their centre of gravity at G .

Hence G must lie on G_1G_2 and be such that

$$W_1 \cdot GG_1 = W_2 \cdot GG_2.$$

Hence, given G and G_1 , we obtain G_2 by producing G_1G to G_2 , so that

$$\begin{aligned} GG_2 &= \frac{W_1}{W_2} \cdot GG_1 \\ &= \frac{W_1}{W - W_1} \cdot GG_1. \end{aligned}$$

The required point may also be obtained by means of Art. 78.

Ex. 1. From a circular disc, of radius r , is cut out a circle, whose diameter is a radius of the disc; find the centre of gravity of the remainder.

Since the areas of circles are to one another as the squares of their radii (App. II), we have

area of the portion cut out : area of the whole circle

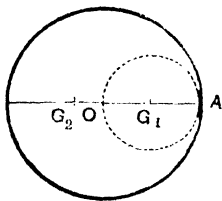
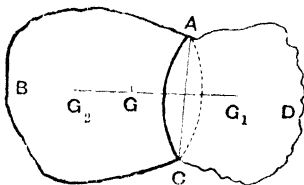
$$\begin{aligned} &:: \left(\frac{r}{2}\right)^2 : r^2 \\ &:: 1 : 4. \end{aligned}$$

Hence the portion cut off is one-quarter, and the portion remaining is three-quarters, of the whole, so that $W_1 = \frac{1}{3}W_2$.

Now the portions W_1 and W_2 make up the whole disc, and therefore balance about O .

Hence

$$\begin{aligned} W_2 \cdot OG_2 &= W_1 \cdot OG_1 = \frac{1}{3}W_2 \times \frac{1}{2}r. \\ \therefore OG_2 &= \frac{1}{6}r. \end{aligned}$$



***Ex. 2.** From a triangular lamina ABC is cut off, by a line parallel to its base BC , one-quarter of its area; find the centre of gravity of the remainder.

Let AB_1C_1 be the portion cut off, so that

$$\triangle AB_1C_1 : \triangle ABC :: 1 : 4.$$

By geometry, since the triangles AB_1C_1 and ABC are similar, we have

$$\triangle AB_1C_1 : \triangle ABC :: AB_1^2 : AB^2.$$

$$\therefore AB_1^2 : AB^2 :: 1 : 4,$$

and hence $AB_1 = \frac{1}{2} AB$.

The line B_1C_1 therefore bisects AB , AC , and AD .

Let G and G_1 be the centres of gravity of the triangles ABC and AB_1C_1 respectively; also let W_1 and W_2 be the respective weights of the portion cut off and the portion remaining, so that $W_2 = 3W_1$.

Since W_2 at G_2 and W_1 at G_1 balance about G , we have, by Art. 78,

$$DG = \frac{W_1 \cdot DG_1 + W_2 \cdot DG_2}{W_1 + W_2} = \frac{DG_1 + 3DG_2}{4} \dots\dots\dots(i).$$

But

$$DG = \frac{1}{3} DA = \frac{2}{3} DD_1,$$

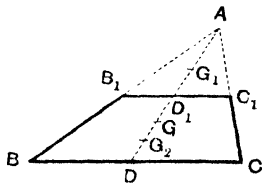
and

$$DG_1 = DD_1 + \frac{1}{3} D_1A = DD_1 + \frac{1}{3} DD_1 = \frac{4}{3} DD_1.$$

Hence (i) is

$$4 \times \frac{2}{3} DD_1 = \frac{4}{3} DD_1 + 3DG_2.$$

$$\therefore DG_2 = \frac{4}{9} DD_1.$$



EXAMPLES. XIII

[Exs. 1, 2, and 4-9 are suitable for verification by **experiment**.]

1. A uniform rod, 1 foot in length, is broken into two parts, of lengths 5 and 7 inches, which are placed so as to form the letter T, the longer portion being vertical; find the centre of gravity of the system.

2. Two rectangular pieces of the same cardboard, of lengths 6 and 8 inches and breadths 2 and $2\frac{1}{2}$ inches respectively, are placed touching, but not overlapping, one another on a table so as to form a T-shaped figure. Find the position of its centre of gravity.

3. A heavy beam consists of two portions, whose lengths are as 3 : 5, and whose weights are as 3 : 1; find the position of its centre of gravity.

4. Two sides of a rectangle are double of the other two, and on one of the longer sides an equilateral triangle is described; find the centre of gravity of the lamina made up of the rectangle and the triangle.

5. A piece of cardboard is in the shape of a square $ABCD$ with an isosceles triangle described on the side BC ; if the side of the square be 12 inches and the height of the triangle be 6 inches, find the distance of the centre of gravity of the cardboard from the line AD .

6. From a parallelogram is cut one of the four portions into which it is divided by its diagonals; find the centre of gravity of the remainder.

7. A parallelogram is divided into four parts, by joining the middle points of opposite sides, and one part is cut away; find the centre of gravity of the remainder.

8. From a square a triangular portion is cut off, by cutting the square along a line joining the middle points of two adjacent sides; find the centre of gravity of the remainder.

9. From a triangle is cut off $\frac{1}{9}$ th of its area by a straight line parallel to its base. Find the position of the centre of gravity of the remainder.

10. A piece of thin uniform wire is bent into the form of a four-sided figure, $ABCD$, of which the sides AB and CD are parallel, and BC and DA are equally inclined to AB . If AB be 18 inches, CD 12 inches, and BC and DA each 5 inches, find the distance from AB of the centre of gravity of the wire.

11. A uniform plate of metal, 10 inches square, has a hole of area 3 square inches cut out of it, the centre of the hole being $2\frac{1}{2}$ inches from the centre of the plate; find the position of the centre of gravity of the remainder of the plate.

12. Where must a circular hole, of 1 foot radius, be punched out of a circular disc, of 3 feet radius, so that the centre of gravity of the remainder may be 2 inches from the centre of the disc?

13. Two uniform spheres, composed of the same materials, and whose diameters are 6 and 12 inches respectively, are firmly united; find the position of their centre of gravity. [The volumes of two spheres are in the ratio of the cubes of their radii.]

14. A solid right circular cone of homogeneous iron, of height 64 inches and mass 8192 lbs., is cut by a plane perpendicular to its axis so that the mass of the small cone removed is 686 lbs. Find the height of the centre of gravity of the truncated portion above the base of the cone.

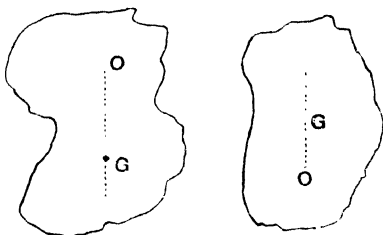
15. The mass of the Moon is .013 times that of the Earth. Taking the Earth's radius as 4000 miles and the distance of the Moon from the centre of the Earth as 60 times the Earth's radius, find the distance of the c.g. of the Earth and Moon from the centre of the Earth.

PROPERTIES OF THE CENTRE OF GRAVITY

85. *If a rigid body be in equilibrium, one point only of the body being fixed, the centre of gravity of the body will be in the vertical line passing through the fixed point of the body.*

Let O be the fixed point of the body, and G its centre of gravity.

The forces acting on the body are the reaction at the fixed point of support of the body, and the weights of the component parts of the body.



The weights of these component parts are equivalent to a single vertical force through the centre of gravity of the body.

Also, when two forces keep a body in equilibrium, they must be equal and opposite and have the same line of action. But the lines of action cannot be the same unless the vertical line through G passes through the point O .

Two cases arise, the first, in which the centre of gravity G is below the point of suspension O , and the second, in which G is above O .

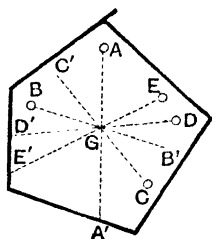
In the first case, the body, if slightly displaced from its position of equilibrium, will tend to return to this position; in the second case, the body will not tend to return to its position of equilibrium.

86. *To find, by experiment, the centre of gravity of a body of any shape.*

Take a flat piece of cardboard of any shape. Bore several small holes $A, B, C, D...$ in it of a size just large enough to freely admit of the insertion of a small pin.

Hang up the cardboard by the hole A and allow it to hang freely and come to rest. Mark on the cardboard the line

AA' which is now vertical. This may be done by hanging from the pin a fine piece of string with a small plummet of lead at the other end, the string having first been well rubbed with chalk. If the string be now flipped against the cardboard it will leave a chalked line, which is AA' . Now hang up the cardboard with the hole B on the pin, and mark in a similar manner the line BB' which is now vertical.

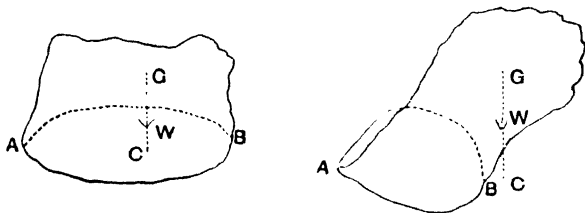


Perform the experiment again with the points C, D, E as the points through which the small pin passes, and obtain the corresponding vertical lines CC', DD', EE' .

These chalked lines AA', BB', CC', DD', EE' will all be found to pass through the same point G . If the thickness of the cardboard be neglected, this point G is its centre of gravity. If the pin be now passed through G , the cardboard will be found to rest in any position in which it is placed.

87. *If a body be placed with its base in contact with a horizontal plane, it will stand, or fall, according as the vertical line drawn through the centre of gravity of the body meets the plane within, or without, the base.*

The forces acting on the body are its weight, which acts at its centre of gravity G , and the reactions of the plane,



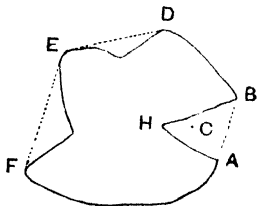
acting at different points of the base of the body. These reactions are all vertical, and hence they may be compounded into a single vertical force acting at some point of the base (Art. 49).

Since the resultant of two *like* parallel forces acts always at a point *between* the forces, it follows that the resultant of

all the pressures on the base of the body cannot act through a point *outside* the base.

Hence, if the vertical line through the centre of gravity of the body meet the plane at a point outside the base, it cannot be balanced by the resultant pressure, and the body cannot therefore be in equilibrium, but must fall over.

If the base of the body has a re-entrant angle, as in the figure, we must extend the meaning of the word "base" in the enunciation to mean the area obtained by drawing a piece of thread tightly round the geometrical base. In the figure the "base" therefore means the area *ABDEFA*.



For example, the point *C*, at which the resultant pressure acts, may lie within the area *AHB*, but it cannot lie without the dotted line *AB*.

If the point *C* were on the line *AB*, between *A* and *B*, the body would be on the point of falling over.

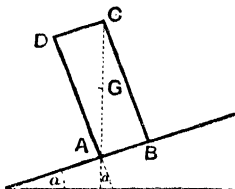
88. Ex. *A cylinder of height h , and the radius of whose base is r , is placed on an inclined plane and prevented from sliding; if the inclination of the plane be gradually increased, find when the cylinder will topple.*

Let the figure represent the section of the cylinder when it is on the point of toppling over; the vertical line through the centre of gravity *G* of the body must therefore just pass through the end *A* of the base. Hence *CAD* must be equal to the angle of inclination, α , of the plane.

Hence

$$\tan \alpha = \tan CAD = \frac{CD}{DA} = \frac{2r}{h},$$

giving the required inclination of the plane.



Stable, unstable, and neutral equilibrium

89. We have pointed out in Art. 85 that the body in the first figure of that article would, if slightly displaced, tend to return to its position of equilibrium, and that the body in the second figure would not tend to return to its

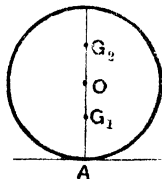
original position of equilibrium, but would recede still further from that position.

These two bodies are said to be in stable and unstable equilibrium respectively.

Again, a cone, resting with its flat circular base in contact with a horizontal plane, would, if slightly displaced, return to its position of equilibrium; if resting with its vertex in contact with the plane it would, if slightly displaced, recede still further from its position of equilibrium; whilst, if placed with its slant side in contact with the plane, it would remain in equilibrium in any position. The equilibrium in the latter case is said to be neutral.

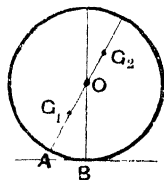
90. Consider, again, the case of a heavy sphere, resting on a horizontal plane, whose centre of gravity is not at its centre.

Let the first figure represent the position of equilibrium, the centre of gravity being either below the centre O , as G_1 , or above, as G_2 . Let the second figure represent the sphere turned through a small angle, so that B is now the point of contact with the plane.



The pressure of the plane still acts through the centre of the sphere.

If the weight of the body acts through G_1 , it is clear that the body will return towards its original position of equilibrium, and therefore the body was originally in stable equilibrium.



If the weight act through G_2 , the body will move still further from its original position of equilibrium, and therefore it was originally in unstable equilibrium.

If however the centre of gravity of the body had been at O , then, in the case of the second figure, the weight would still be balanced by the pressure of the plane; the body would thus remain in the new position, and the equilibrium would be called neutral.

91. Def. A body is said to be in **stable** equilibrium when, if it be slightly displaced from its position of equilibrium, the forces acting on the body tend to make it

return towards its position of equilibrium; it is in **unstable** equilibrium when, if it be slightly displaced, the forces tend to move it still further from its position of equilibrium; it is in **neutral** equilibrium, if the forces acting on it in its displaced position be in equilibrium.

In general bodies which are "top-heavy," or which have small bases, are unstable.

Thus in theory a pin might be placed upright with its point on a horizontal table so as to be in equilibrium; in practice the "base" would be so small that the slightest displacement would bring the vertical through its centre of gravity outside its base and it would fall. So with a billiard cue placed vertically with its end on the table.

A body is, as a general principle, in a stable position of equilibrium when the centre of gravity is in the lowest position it can take up; examples are the case of the last article, and the pendulum of a clock; the latter when displaced always returns towards its position of rest.

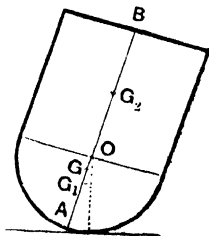
Consider again the case of a man walking on a tight rope. He always carries a pole heavily weighted at one end, so that the centre of gravity of himself and the pole is always below his feet. When he feels himself falling in one direction, he shifts his pole so that this centre of gravity shall be on the other side of his feet, and then the resultant weight pulls him back again towards the upright position.

If a body has more than one theoretical position of equilibrium, the one in which its centre of gravity is lowest will in general be the stable position, and that in which the centre of gravity is highest will be the unstable one.

***92. Ex.** *A homogeneous body, consisting of a cylinder and a hemisphere joined at their bases, is placed with the hemispherical end on a horizontal table; is the equilibrium stable or unstable?*

Let G_1 and G_2 be the centres of gravity of the hemisphere and cylinder, let A be the point of the body which is initially in contact with the table, and let O be the centre of the base of the hemisphere.

If h be the height of the cylinder, and r be



the radius of the base, we have

$$OG_1 = \frac{3}{8}r \text{ (Art. 77), and } OG_2 = \frac{h}{2}.$$

Also the weights of the hemisphere and cylinder are proportional to $\frac{2}{3}\pi r^3$ and $\pi \cdot r^2 h$. (App. II.)

The reaction of the plane, in the displaced position of the body, always passes through the centre O .

The equilibrium is stable or unstable according as G , the centre of gravity of the compound body, is below or above O , *i.e.*, according as

$$OG_1 \times \text{wt. of hemisphere is } \gtrless OG_2 \times \text{wt. of cylinder,}$$

$$\textit{i.e.}, \text{ according as } \frac{3}{8}r \times \frac{2}{3}\pi r^3 \text{ is } \gtrless \frac{h}{2} \times \pi r^2 h,$$

$$\textit{i.e.}, \text{ according as } \frac{r^2}{2} \text{ is } \gtrless h^2,$$

$$\textit{i.e.}, \text{ according as } r \text{ is } \gtrless \sqrt{2}h,$$

$$\textit{i.e.} \qquad \qquad \qquad \gtrless h \times 1.42 \dots$$

EXAMPLES. XIV

1. A carpenter's rule, 2 feet in length, is bent into two parts at right angles to one another, the length of the shorter portion being 8 inches. If the shorter be placed on a smooth horizontal table, what is the length of the least portion on the table that there may be equilibrium?

2. A cylinder, whose base is a circle of one foot diameter and whose height is 3 feet, rests on a horizontal plane with its axis vertical. Find how high one edge of the base can be raised before the cylinder overturns.

3. A hollow vertical cylinder, of radius $2a$ and height $3a$, rests on a horizontal table, and a rod is placed within it with its lower end resting on the circumference of the base; if the weight of the rod be equal to that of the cylinder, how long must the rod be so that it may just cause the cylinder to topple over?

4. A square table stands on four legs placed respectively at the middle points of its sides; find the greatest weight that can be put at one of the corners without upsetting the table.

5. A square four-legged table has lost one leg; where on the table should a weight, equal to the weight of the table, be placed, so that the pressures on the three remaining legs of the table may be equal?

6. A square table, of weight 20 lbs., has legs at the middle points of its sides, and three equal weights, each equal to the weight of the table, are placed at three of the angular points. What is the greatest weight that can be placed at the fourth corner so that equilibrium may be preserved?

7. The side CD of a uniform square plate $ABCD$, whose weight is W , is bisected at E and the triangle AED is cut off. The plate $ABCEA$ is placed in a vertical position with the side CE on a horizontal plane. What is the greatest weight that can be placed at A without upsetting the plate?

8. ABC is a flat board, A being a right angle and AC in contact with a flat table; D is the middle point of AC and the triangle ABD is cut away; shew that the triangle is just on the point of falling over.

9. ABC is an isosceles triangle, of weight W , of which the angle A is 120° , and the side AB rests on a smooth horizontal table, the plane of the triangle being vertical; if a weight $\frac{W}{3}$ be hung on at C , shew that the triangle will just be on the point of toppling over.

10. A number of bricks, each 9 inches long, 4 inches wide, and 3 inches thick, are placed one on another so that, whilst their narrowest surfaces, or thicknesses, are in the same vertical plane, each brick overlaps the one underneath it by half an inch; the lowest brick being placed on a table, how many bricks can be so placed without their falling over?

11. A brick is laid with one-quarter of its length projecting over the edge of a wall; a brick and one-quarter of a brick are then laid on the first with one-quarter of a brick projecting over the edge of the first brick; a brick and a half are laid on this, and so on; shew that 4 courses of brick laid in the above manner will be in equilibrium without the aid of mortar, but that, if a fifth course be added, the structure will topple.

12. A pile of six pennies rests on a horizontal table, and each penny projects the same distance beyond the one below it. Find the greatest possible horizontal distance between the centres of the highest and lowest pennies.

13. A solid uniform hemisphere rests upon a horizontal surface with its flat surface horizontal and uppermost. Shew that it is in stable equilibrium.

CHAPTER IX

MACHINES

93. IN the following chapter we shall explain and discuss the equilibrium of some of the simpler machines, viz., (1) The Lever, (2) The Pulley and Systems of Pulleys, (3) The Inclined Plane, (4) The Wheel and Axle, (5) The Common Balance, and (6) The Steelyards.

We shall suppose the different portions of these machines to be smooth and rigid, that all cords or strings used are perfectly flexible, and that the forces acting on them always balance, so that the machines are at rest.

In actual practice these conditions are not even approximately satisfied in the cases of many machines.

94. When two external forces applied to a machine balance, one may be, and formerly always was, called the Power and the other may be called the Weight.

A machine is always used in practice to overcome some resistance; the force we exert on the machine is the power; the resistance to be overcome, in whatever form it may appear, is called the Weight.

Unfortunately the word Power is also used in a different sense with regard to a Machine (Art. 146); of late years the word Effort has been used to denote what was formerly called the Power in the sense of this article. The word Resistance is also used instead of Weight.

95. Mechanical Advantage. If in any machine an effort P balance a resistance W , the ratio $W : P$ is called the mechanical advantage of the machine, so that

$$\text{Resistance} = \text{Effort} \times \text{Mechanical Advantage.}$$

Almost all machines are constructed so that the mechanical advantage is a ratio greater than unity.

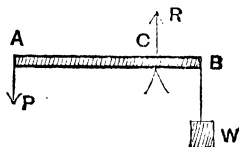
If in any machine the mechanical advantage be less than unity, it may, with more accuracy, be called mechanical disadvantage.

I. The Lever

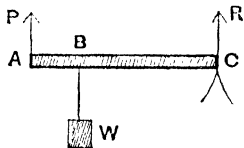
96. The Lever consists essentially of a rigid bar, straight or bent, which has one point fixed about which the rest of the lever can turn. This fixed point is called the Fulcrum, and the perpendicular distances between the fulcrum and the lines of action of the effort and the weight are called the arms of the lever.

When the lever is straight, and the effort and weight act perpendicular to the lever, it is usual to distinguish three classes or orders.

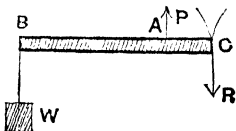
Class I. Here the effort P and the weight W act on opposite sides of the fulcrum C .



Class II. Here the effort P and the weight W act on the same side of the fulcrum C , but the former acts at a greater distance than the latter from the fulcrum.



Class III. Here the effort P and the weight W act on the same side of the fulcrum C , but the former acts at a less distance than the latter from the fulcrum.



97. *Conditions of equilibrium of a straight lever.*

In each case we have three parallel forces acting on the body, so that the reaction, R , at the fulcrum must be equal and opposite to the resultant of P and W .

In the first class P and W are like parallel forces, so that their resultant is $P + W$. Hence

$$R = P + W.$$

In the second class P and W are unlike parallel forces, so that $P + R = W$, i.e. $R = W - P$.

So in the third class $R + W = P$, i.e. $R = P - W$.

In the first and third classes we see that R and P act in opposite directions; in the second class they act in the same direction.

In all three classes, since the resultant of P and W passes through C , we have, as in Art. 47,

$$P \cdot AC = W \cdot BC,$$

i.e. $P \times \text{the arm of } P = W \times \text{the arm of } W$.

Since $\frac{W}{P} = \frac{\text{arm of } P}{\text{arm of } W}$, we observe that generally in Class I, and always in Class II, there is mechanical advantage, but that in Class III there is mechanical disadvantage.

The practical use of levers of the latter class is to apply a force at some point at which it is not easy to apply the force directly.

The principle of the lever was known to Archimedes who lived in the third century B.C.; until the discovery of the Parallelogram of Forces in the sixteenth century it was the fundamental principle of Statics.

98. Examples of the different classes of levers are:

Class I. A Poker (*when used to stir the fire, the bar of the grate being the fulcrum*); A Claw-hammer (*when used to extract nails*); A Crowbar (*when used with a point in it resting on a fixed support*); A Pair of Scales; The Brake of a Pump.

Double levers of this class are: A Pair of Scissors, A Pair of Pincers.

Class II. A Wheelbarrow; A Cork Squeezer; A Crowbar (*with one end in contact with the ground*); An Oar (*assuming the end of the oar in contact with the water to be at rest*).

A Pair of Nutcrackers is a double lever of this class.

Class III. The Treadle of a Lathe; The Human Fore-arm (*when the latter is used to support a weight placed on the palm of the hand. The Fulcrum is the elbow, and the tension exerted by the muscles is the effort*).

A Pair of Sugar-tongs is a double lever of this class.

99. In Art. 97 we have neglected the weight of the bar itself. If the weight be taken into consideration, or if the lever be bent, we must obtain the conditions of equilibrium by equating to zero the algebraic sum of the moments of the forces about the fulcrum.

Ex. If two weights balance, about a fixed fulcrum, at the extremities of a straight lever, in any position inclined to the vertical, they will balance in any other position.

Let AB be the lever, of weight W' , and let its centre of gravity be G . Let the lever balance about a fulcrum O in any position inclined at an angle θ to the horizontal, the weights at A and B being P and W respectively.

Through O draw a horizontal line $LONM$ to meet the lines of action of P , W' , and W in L , N , and M respectively.

Since the forces balance about O , we have

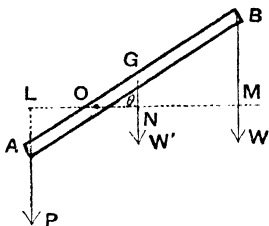
$$P \cdot OL = W \cdot OM + W' \cdot ON.$$

$$\therefore P \cdot OA \cos \theta = W \cdot OB \cos \theta + W' \cdot OG \cos \theta.$$

$$\therefore P \cdot OA = W \cdot OB + W' \cdot OG.$$

This condition of equilibrium is independent of the inclination θ of the lever to the horizontal; hence in any other position of the lever the condition would be the same.

Hence, if the lever be in equilibrium in one position, it will be in equilibrium in all positions.



EXAMPLES. XV

1. In a weightless lever, if one of the forces be equal to 10 lbs. wt. and the pressure on the fulcrum be equal to 16 lbs. wt., and the length of the shorter arm be 3 feet, find the length of the longer arm.

2. Where must the fulcrum be so that a weight of 6 lbs. may balance a weight of 8 lbs. on a straight weightless lever, 7 feet long?

If each weight be increased by 1 lb., in what direction will the lever turn?

3. If two forces, applied to a weightless lever, balance, and if the pressure on the fulcrum be ten times the difference of the forces, find the ratio of the arms.

4. A lever, 1 yard long, has weights of 6 and 20 lbs. fastened to its ends, and balances about a point distant 9 inches from one end; find its weight.

5. A straight lever, AB , 12 feet long, balances about a point, 1 foot from A , when a weight of 18 lbs. is suspended from A . It will balance

about a point, which is 1 foot from *B*, when a weight of 11 lbs. is suspended from *B*. Shew that the centre of gravity of the lever is 5 inches from the middle point of the lever.

6. A uniform lever is 18 inches long and is of weight 18 ounces; find the position of the fulcrum when a weight of 27 ounces at one end of the lever balances one of 9 ounces at the other.

If the lesser weight be doubled, by how much must the position of the fulcrum be shifted so as to preserve equilibrium?

7. The short arm of one lever is hinged to the long arm of a second lever, and the short arm of the latter is attached to a press; the long arms being each 3 feet in length, and the short arms 6 inches, find what pressure will be produced on the press by a force, equal to 10 stone weight, applied to the end of the long arm of the first lever.

8. The arms of a bent lever are at right angles to one another, and they are in the ratio of 5 to 1. The longer arm is inclined to the horizon at an angle of 45° , and carries at its end a weight of 10 lbs.; the end of the shorter arm presses against a horizontal plane; find the pressure on the plane.

9. Shew that the propelling force on an eight-oared boat is 224 lbs. weight, supposing each man to pull his oar with a force of 56 lbs. weight, and that the length of the oar from the middle of the blade to the handle is three times that from the handle to the rowlock.

10. In a pair of nutcrackers, 5 inches long, if the nut be placed at a distance of $\frac{7}{8}$ inch from the hinge, a pressure of $3\frac{1}{2}$ lbs. applied to the ends of the arms will crack the nut. What weight placed on the top of the nut will crack it?

11. A man raises a 3-foot cube of stone, weighing 2 tons, by means of a crowbar, 4 feet long, after having thrust one end of the bar under the stone to a distance of 6 inches; what force must be applied at the other end of the bar to raise the stone?

II. Pulleys

100. A pulley is composed of a wheel of wood, or metal, grooved along its circumference to receive a string or rope; it can turn freely about an axle passing through its centre perpendicular to its plane, the ends of this axle being supported by a frame of wood called the block.

A pulley is said to be movable or fixed according as its block is movable or fixed.

The weight of the pulley is often so small, compared with the weights which it supports, that it may be neglected; such a pulley is called a weightless pulley.

We shall always neglect the weight of the string or rope which passes round the pulley.

We shall also always consider the pulley to be perfectly smooth, so that the tension of a string which passes round a pulley is constant throughout its length.

101. Single Pulley. The use of a single pulley is to apply an effort in a different direction from that in which it is convenient to us to apply the effort.

Thus, in the first figure, a man standing on the ground and pulling vertically at one end of the rope might support a weight W hanging at the other end; in the second figure the same man pulling sideways might support the weight.

In each case the tension of the string passing round the pulley is unaltered; the effort P is therefore equal to the weight W .

In the first figure the action on the fixed support to which the block is attached must balance the other forces on the pulley-block, and must therefore be equal to

$$W + P + w,$$

i.e., $2W + w$, where w is the weight of the pulley-block.

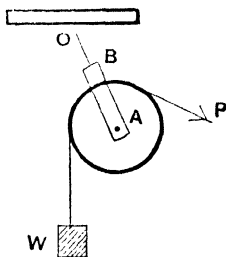
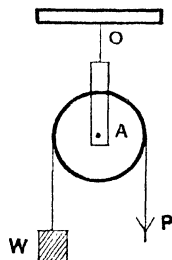
In the second figure, neglecting the weight of the pulley, the effort P , and the weight W , being equal, must be equally inclined to the line OA .

Hence, if T be the tension of the supporting string OB and 2θ the angle between the directions of P and W , we have

$$T = P \cos \theta + W \cos \theta = 2W \cos \theta.$$

102. We shall discuss three systems of pulleys and shall follow the usual order; there is no particular reason for this order, but it is convenient to retain it for purposes of reference.

First system of Pulleys. Each string attached to the



supporting beam. To find the relation between the effort or "power" and the weight.

In this system of pulleys the weight is attached to the lowest pulley, and the string passing round it has one end attached to the fixed beam, and the other end attached to the next highest pulley; the string passing round the latter pulley has one end attached to the fixed beam, and the other to the next pulley, and so on; the effort is applied to the free end of the last string.

Often there is an additional fixed pulley over which the free end of the last string passes; the effort may then be applied as a downward force.

Let $A_1, A_2, \dots$ be the pulleys, beginning from the lowest, and let the tensions of the strings passing round them be $T_1, T_2, \dots$. Let W be the weight and P the effort.

[N.B. The string passing round any pulley, A_2 say, pulls A_2 vertically upwards, and pulls A_3 downwards.]

I. Let the weights of the pulleys be neglected.

From the equilibrium of the pulleys $A_1, A_2, \dots$, taken in order, we have

$$2T_1 = W; \therefore T_1 = \frac{1}{2}W.$$

$$2T_2 = T_1; \therefore T_2 = \frac{1}{2}T_1 = \frac{1}{2^2}W.$$

$$2T_3 = T_2; \therefore T_3 = \frac{1}{2}T_2 = \frac{1}{2^3}W.$$

$$2T_4 = T_3; \therefore T_4 = \frac{1}{2}T_3 = \frac{1}{2^4}W.$$

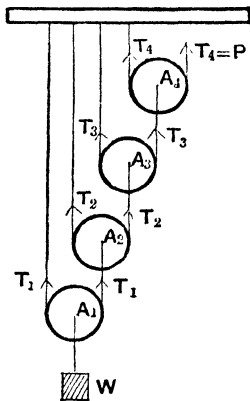
But, with our figure, $T_4 = P$. $\therefore P = \frac{1}{2^4}W$.

Similarly, if there were n pulleys, we should have

$$P = \frac{1}{2^n}W.$$

Hence, in this system of pulleys, the mechanical advantage

$$= \frac{W}{P} = 2^n.$$



II. Let the weights of the pulleys in succession, beginning from the lowest, be $w_1, w_2, \dots$

In this case we have an additional downward force on each pulley.

As before, we have

$$2T_1 = W + w_1,$$

$$2T_2 = T_1 + w_2,$$

$$2T_3 = T_2 + w_3,$$

$$2T_4 = T_3 + w_4.$$

$$\therefore T_1 = \frac{W}{2} + \frac{w_1}{2},$$

$$T_2 = \frac{1}{2}T_1 + \frac{w_2}{2} = \frac{W}{2^2} + \frac{w_1}{2^2} + \frac{w_2}{2},$$

$$T_3 = \frac{1}{2}T_2 + \frac{w_3}{2} = \frac{W}{2^3} + \frac{w_1}{2^3} + \frac{w_2}{2^2} + \frac{w_3}{2},$$

and $P = T_4 = \frac{1}{2}T_3 + \frac{w_4}{2} = \frac{W}{2^4} + \frac{w_1}{2^4} + \frac{w_2}{2^3} + \frac{w_3}{2^2} + \frac{w_4}{2}.$

Similarly, if there were n pulleys, we should have

$$P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \dots + \frac{w_n}{2} \dots \dots \dots (1).$$

It follows that the mechanical advantage, $\frac{W}{P}$, depends on the weight of the pulleys.

In this system of pulleys we observe that the greater the weight of the pulleys, the greater must P be to support a given weight W ; the weights of the pulleys oppose the effort, and the pulleys should therefore be made as light as is consistent with the required strength.

Stress on the beam from which the pulleys are hung.

Let R be the stress on the beam. Since R , together with the effort P , supports the system of pulleys, together with the weight W , we have

$$R + P = W + w_1 + w_2 + \dots + w_n.$$

From this equation and (1) we easily obtain R .

Ex. If there be 4 movable pulleys, whose weights, commencing with the lowest, are 4, 5, 6, and 7 lbs., what effort will support a body of weight 1 cwt., and what is the stress on the beam?

Using the notation of the previous article, we have

$$2T_1 = 112 + 4; \quad \therefore T_1 = 58.$$

$$2T_2 = T_1 + 5 = 63; \quad \therefore T_2 = 31\frac{1}{2}.$$

$$2T_3 = T_2 + 6 = 37\frac{1}{2}; \quad \therefore T_3 = 18\frac{3}{4}.$$

$$2P = T_3 + 7 = 25\frac{3}{4}; \quad \therefore P = 12\frac{7}{8} \text{ lbs. wt.}$$

Also

$$R + P = 112 + 4 + 5 + 6 + 7 = 134.$$

$$\therefore R = 121\frac{1}{8} \text{ lbs. wt.}$$

EXAMPLES. XVI

1. In the following cases, the movable pulleys are weightless, their number is n , the weight is W , and the "power" or effort is P :

(1) If $n=4$ and $P=20$ lbs. wt., find W ;

(2) If $n=4$ and $W=1$ cwt., find P ;

(3) If $W=56$ lbs. wt. and $P=7$ lbs. wt., find n .

2. In the following cases, the movable pulleys are of equal weight w , and are n in number, P is the effort, and W is the weight:

(1) If $n=4$, $w=1$ lb. wt., and $W=97$ lbs. wt., find P

(2) If $n=3$, $w=1\frac{1}{3}$ lbs. wt., and $P=7$ lbs. wt., find W ;

(3) If $n=5$, $W=775$ lbs. wt., and $P=31$ lbs. wt., find w ;

(4) If $W=107$ lbs. wt., $P=2$ lbs. wt., and $w=\frac{1}{3}$ lb. wt., find n .

3. In the first system of pulleys, if there be 4 pulleys, each of weight 2 lbs., what weight can be raised by an effort equal to the weight of 20 lbs.?

4. If there be 3 movable pulleys, whose weights, commencing with the lowest, are 9, 2, and 1 lbs. respectively, what effort will support a weight of 69 lbs.?

5. If there be 4 movable pulleys, whose weights, commencing with the lowest, are 4, 3, 2, and 1 lbs. respectively, what effort will support a weight of 54 lbs.?

6. If there be 3 movable pulleys and their weights beginning from the lowest be 4, 2, and 1 lbs. respectively, what effort will be required to support a weight of 28 lbs.?

7. A system consists of 4 pulleys, arranged so that each hangs by a separate string, one end being fastened to the upper block, and all the free ends being vertical. If the weights of the pulleys, beginning at the lowest, be w , $2w$, $3w$, and $4w$, find the effort necessary to support a weight $15w$, and the magnitude of the single force necessary to support the beam to which the other ends of the strings are attached.

8. A man, of 12 stone weight, is suspended from the lowest of a system of 4 weightless pulleys, in which each hangs by a separate string, and supports himself by pulling at the end of the string which passes over a fixed pulley. Find the amount of his pull on this string.

9. A man, whose weight is 156 lbs., is suspended from the lowest of a system of 4 pulleys, each being of weight 10 lbs., and supports himself by pulling at the end of the string which passes over the fixed pulley. Find the force which he exerts on the string, supposing all the strings to be vertical.

103. Second system of pulleys. *The same string passing round all the pulleys. To find the relation between the effort and the weight.*

In this system there are two blocks, each containing pulleys, the upper block being fixed and the lower block movable. The same string passes round all the pulleys as in the figures.

If the number of pulleys in the upper block be the same as in the lower block (Fig. 1), one end of the string must be fastened to the upper block; if the number in the upper block be greater by one than the number in the lower block (Fig. 2), the end of the string must be attached to the lower block.

In the first case, the number of portions of string connecting the blocks is even; in the second case, the number is odd.

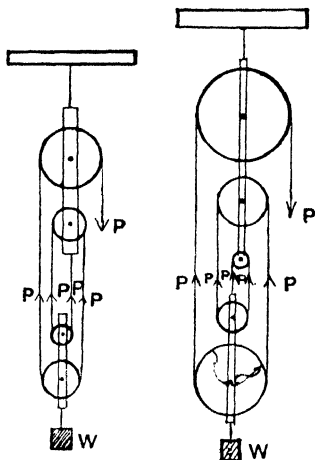


Fig. 1

Fig. 2

In either case, let n be the number of portions of string at the lower block. Since we have only one string passing over smooth pulleys, the tension of each of these portions is P , so that the total upward force at the lower block is $n \cdot P$.

Let W be the weight supported, and w the weight of the lower block.

Hence $W + w = nP$, giving the relation required.

In practice the pulleys of each block are often placed parallel to one another, so that the strings are not mathematically parallel; they are, however, very approximately parallel, so that the above relation is still very approximately true.

EXAMPLES. XVII

1. If a weight of 5 lbs. support a weight of 24 lbs., find the weight of the lower block, when there are 3 pulleys in each block.

2. If weights of 5 and 6 lbs. respectively at the free ends of the string support weights of 18 and 22 lbs. at the lower block, find the number of the strings and the weight of the lower block.

3. If weights of 4 lbs. and 5 lbs. support weights of 5 lbs. and 18 lbs. respectively, what is the weight of the lower block, and how many pulleys are there in it?

4. A weight of 6 lbs. just supports a weight of 28 lbs., and a weight of 8 lbs. just supports a weight of 42 lbs.; find the number of strings and the weight of the lower block.

5. In the second system of pulleys, if a basket be suspended from the lower block and a man in the basket support himself and the basket, by pulling at the free end of the string, find the tension he exerts, neglecting the inclination of the string to the vertical, and assuming the weight of the man and basket to be W .

6. A man, whose weight is 12 stone, raises 3 cwt. by means of a system of pulleys in which the same string passes round all the pulleys, there being 4 in each block, and the string being attached to the upper block; neglecting the weights of the pulleys, find what will be his pressure on the ground if he pull vertically downwards.

7. We are told that the cable by which "Great Paul," whose weight is 18 tons, was lifted into its place in the Cathedral tower, passed four times through the two blocks of pulleys. From this statement give a description of the pulleys, and estimate the strength of the cable.

8. An ordinary block and tackle has two pulleys in the lower block and two in the upper. What force must be exerted to lift a load of 300 lbs.? If on account of friction a given force will only lift $\cdot 45$ times as much as if the system were frictionless, find the force required.

104. Third system of pulleys. *All the strings attached to the weight. To find the relation between the effort and the weight.*

In this system the string passing round any pulley is attached at one end to a bar, from which the weight is suspended, and at the other end to the next lower pulley; the string round the lowest pulley is attached at one end to the bar, whilst at the other end of this string the effort is applied.

In this system the upper pulley is fixed.

Let $A_1, A_2, A_3 \dots$ be the movable pulleys, beginning from

the lowest, and let the tensions of the strings passing round these pulleys respectively be

$$T_1, T_2, T_3 \dots$$

If the effort be P , we have clearly

$$T_1 = P.$$

I. *Let the weights of the pulleys be neglected.*

For the equilibrium of the pulleys, taken in order and commencing from the lowest, we have

$$T_2 = 2T_1 = 2P,$$

$$T_3 = 2T_2 = 2^2P,$$

and

$$T_4 = 2T_3 = 2^3P.$$

But, since the bar, from which W is suspended, is in equilibrium, we have

$$\begin{aligned} W &= T_1 + T_2 + T_3 + T_4 \\ &= P + 2P + 2^2P + 2^3P \\ &= P \frac{2^4 - 1}{2 - 1} = P(2^4 - 1) \dots (1). \end{aligned}$$

If there were n pulleys, of which $(n - 1)$ would be movable, we should have, similarly,

$$\begin{aligned} W &= T_1 + T_2 + T_3 + \dots + T_n \\ &= P + 2P + 2^2P + \dots + 2^{n-1}P \\ &= P \left[\frac{2^n - 1}{2 - 1} \right], \end{aligned}$$

by summing the geometrical progression,

$$= P(2^n - 1) \dots (2).$$

Hence the mechanical advantage is $2^n - 1$.

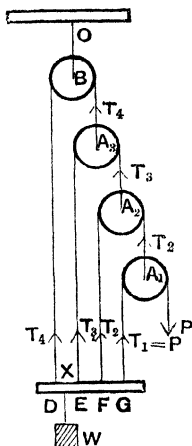
II. *Let the weights of the movable pulleys, taken in order and commencing with the lowest, be $w_1, w_2 \dots$*

Considering the equilibrium of the pulleys in order, we have

$$T_2 = 2T_1 + w_1 = 2P + w_1,$$

$$T_3 = 2T_2 + w_2 = 2^2P + 2w_1 + w_2,$$

$$T_4 = 2T_3 + w_3 = 2^3P + 2^2w_1 + 2w_2 + w_3.$$



But, from the equilibrium of the bar,

$$\begin{aligned} W &= T_4 + T_3 + T_2 + T_1 \\ &= (2^3 + 2^2 + 2 + 1)P + (2^2 + 2 + 1)w_1 + (2 + 1)w_2 + w_3 \\ &= \frac{2^4 - 1}{2 - 1}P + \frac{2^3 - 1}{2 - 1}w_1 + \frac{2^2 - 1}{2 - 1}w_2 + w_3 \\ &= (2^4 - 1)P + (2^3 - 1)w_1 + (2^2 - 1)w_2 + w_3 \dots\dots\dots(3). \end{aligned}$$

If there were n pulleys, of which $(n-1)$ would be movable, we should have, similarly,

$$\begin{aligned} W &= (2^n - 1)P + (2^{n-1} - 1)w_1 + (2^{n-2} - 1)w_2 + \dots \\ &\quad + (2^2 - 1)w_{n-2} + (2 - 1)w_{n-1} \dots\dots\dots(4). \end{aligned}$$

Stress on the supporting beam. This stress balances the effort, the weight, and the weight of the pulleys, and therefore equals

$$P + W + w_1 + w_2 + \dots + w_n,$$

and hence is easily found.

Ex. If there be 4 pulleys, whose weights, commencing with the lowest, are 4, 5, 6, and 7 lbs., what effort will support a body of weight 1 cwt.?

Using the notation of the previous article, we have

$$\begin{aligned} T_2 &= 2P + 4, \\ T_3 &= 2T_2 + 5 = 4P + 13, \\ T_4 &= 2T_3 + 6 = 8P + 32. \end{aligned}$$

$$\text{Also} \quad 112 = T_4 + T_3 + T_2 + P = 15P + 49.$$

$$\therefore P = \frac{63}{15} = 4\frac{1}{5} \text{ lbs. wt.}$$

*105. In this system we observe that, the greater the weight of each pulley, the less is P required to be in order that it may support a given weight W . Hence the weights of the pulleys assist the effort.

If the weights of the pulleys be properly chosen, the system will remain in equilibrium without the application of any effort whatever.

For example, suppose we have 3 movable pulleys, each of weight w , the relation (3) of the last article will become

$$W = 15P + 11w.$$

Hence, if $11w = W$, we have P zero, *i.e.*, if the weight to be supported be eleven times the weight of each of the three movable pulleys, no effort need be applied at the free end of the string to preserve equilibrium.

*106. In the third system of pulleys, the bar supporting the weight W will not remain horizontal, unless the point at which the weight is attached be properly chosen.

In any particular case the proper point of attachment can be easily found.

Taking the figure of Art. 104 let there be three movable pulleys,

whose weights are negligible. Let the distances between the points D , E , F , and G at which the strings are attached, be successively a , and let the point at which the weight is attached be X .

The resultant of T_1 , T_2 , T_3 , and T_4 must pass through X .

Hence by Art. 78,

$$\begin{aligned} DX &= \frac{T_4 \times 0 + T_3 \times a + T_2 \times 2a + T_1 \times 3a}{T_4 + T_3 + T_2 + T_1} \\ &= \frac{4P \cdot a + 2P \cdot 2a + P \cdot 3a}{8P + 4P + 2P + P} = \frac{11a}{15} \end{aligned}$$

$\therefore DX = \frac{11}{15} DE$, giving the position of X .

This system of pulleys is not however designed in order to lift weights. If it were used for this purpose it would soon be found to be unworkable. Its use is to give a short strong pull. For example, it is used on board a yacht to set up the back stay.

In the figure of Art. 104, $DEFG$ is the deck of the yacht to which the strings are attached, and there is no W . The strings to the pulleys A_1 , A_2 , A_3 , and A_4 are inclined to the vertical, and the point O is at the top of the mast which is to be kept erect. The resistance in this case is the force at O necessary to keep the mast up, and the effort is applied as in the figure.

EXAMPLES. XVIII

1. In the following cases, the pulleys are weightless and n in number, P is the "power" or effort, and W the weight:

- (1) If $n=4$ and $P=2$ lbs. wt., find W ;
- (2) If $n=5$ and $W=124$ lbs. wt., find P ;
- (3) If $W=105$ lbs. and $P=7$ lbs. wt., find n .

2. In the following cases, the pulleys are equal and each of weight w , P is the effort, and W is the weight:

- (1) If $n=4$, $w=1$ lb. wt., and $P=10$ lbs. wt., find W ;
- (2) If $n=3$, $w=\frac{1}{2}$ lb. wt., and $W=114$ lbs. wt., find P ;
- (3) If $n=5$, $P=3$ lbs. wt., and $W=106$ lbs. wt., find w ;
- (4) If $P=4$ lbs. wt., $W=137$ lbs. wt., and $w=\frac{1}{2}$ lb. wt., find n .

3. If there be 5 pulleys, each of weight 1 lb., what effort is required to support 3 cwt.?

If the pulleys be of equal size, find to what point of the bar the weight must be attached, so that the beam may be always horizontal.

4. If the strings passing round a system of 4 weightless pulleys be fastened to a rod without weight at distances successively an inch apart, find to what point of the rod the weight must be attached, so that the rod may be always horizontal.

5. Find the mechanical advantage, when the pulleys are 4 in number, and each is of weight $\frac{1}{64}$ th that of the weight.

6. In a system of 3 weightless pulleys, in which each string is attached to a bar which carries the weight, if the diameter of each pulley be 2 inches, find to what point of the bar the weight should be attached so that the bar may be always horizontal.

7. In the third system of 3 pulleys, if the weights of the pulleys be all equal, find the relation of the effort to the weight when equilibrium is established. If each pulley weigh 2 ounces, what weight would be supported by the pulleys only?

If the weight supported be 25 lbs. wt., and the power be 3 lbs. wt., find what must be the weight of each pulley.

8. In the third system of weightless pulleys, the weight is supported by a power of 70 lbs. The hook by which one of the strings is attached to the weight breaks, and the string is then attached to the pulley which it passed over, and an effort of 150 lbs. is now required. Find the number of pulleys and the weight supported.

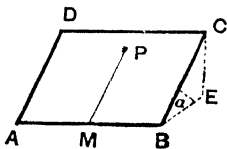
III. The Inclined Plane

107. The Inclined Plane, considered as a mechanical power, is a rigid plane inclined at an angle to the horizon.

It is used to facilitate the raising of heavy bodies.

In the present chapter we shall only consider the case of a body resting on the plane, and acted upon by forces in a plane perpendicular to the intersection of the inclined plane and the horizontal, *i.e.*, in a vertical plane through the line of greatest slope.

The reader can picture to himself the line of greatest slope on an inclined plane in the following manner: take a rectangular sheet of cardboard $ABCD$, and place it at an angle to the horizontal, so that the line AB is in contact with a horizontal table: take any point P on the cardboard and draw PM perpendicular to the line AB ; PM is the line of greatest slope passing through the point P .



From C draw CE perpendicular to the horizontal plane through AB , and join BE . The lines BC , BE , and CE are called respectively the length, base, and height of the inclined plane; also the angle CBE is the inclination of the plane to the horizon.

108. The inclined plane is supposed to be smooth, so that the only reaction between it and any body resting on it is perpendicular to the inclined plane.

Since the plane is rigid, it is capable of exerting any reaction, however great, that may be necessary to give equilibrium.

109. A body, of given weight, rests on a smooth inclined plane; to determine the relations between the effort, the weight, and the reaction of the plane.

Let W be the weight of the body, P the effort, and R the reaction of the plane; also let α be the inclination of the plane to the horizon.

Case I. Let the effort act up the plane along the line of greatest slope.

Let AC be the inclined plane, AB the horizontal line through A , DE a vertical line, and let the perpendicular to the plane through D meet AB in F , and the vertical line through C in the point G .

Hence $\angle FGC = \angle FDE = \alpha$.

Also the right angles GDC and ABC are equal.

Hence the triangles GDC and ABC are equiangular, so that $DC : DG : GC :: BC : AB : AC$ (App. I, Art. 2).

Now the forces P , R , and W are parallel to the sides DC , DG , and GC of the triangle DGC and are therefore proportional to them,

$$\begin{aligned} \therefore P : R : W &:: DC : DG : GC \\ &:: BC : AB : AC \end{aligned}$$

$\therefore$ Height of Plane : Base of Plane : Length of Plane.

Otherwise thus: Resolve W along and perpendicular to the plane; its components are

$W \cos ADE$, i.e., $W \sin \alpha$, along DA ,

and $W \sin ADE$, i.e., $W \cos \alpha$, along DF .

Hence $P = W \sin \alpha$, and $R = W \cos \alpha$.

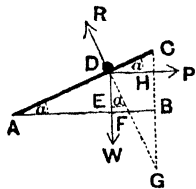
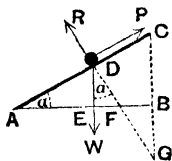
Case II. Let the effort act horizontally.

[In this case we must imagine a small hole in the plane at D through which a string is passed and attached to the body, or else that the body is pushed toward the plane by a horizontal force.]

Let the direction of P meet the vertical line through C in the point H .

As in Case I the triangles GDH and ABC are equiangular,

so that $HD : DG : GH :: BC : CA : AB$ (App. I, Art. 2).



But the forces P , R , and W are parallel to the sides DH , GD , and HG of the triangle HDG and are therefore proportional to them.

$$\therefore P : R : W :: DH : GD : HG \\ :: BC : CA : AB$$

$\therefore$ Height of Plane : Length of Plane : Base of Plane.

Otherwise thus: The components of W along and perpendicular to the plane are $W \sin \alpha$ and $W \cos \alpha$; the components of P , similarly, are $P \cos \alpha$ and $P \sin \alpha$.

$$\therefore P \cos \alpha = W \sin \alpha,$$

and

$$R = P \sin \alpha + W \cos \alpha = W \left[\frac{\sin^2 \alpha}{\cos \alpha} + \cos \alpha \right] = W \frac{\sin^2 \alpha + \cos^2 \alpha}{\cos \alpha} = W \sec \alpha.$$

$$\therefore P = W \tan \alpha, \text{ and } R = W \sec \alpha.$$

***Case III.** Let the effort act at an angle θ with the inclined plane.

By Lami's Theorem we have

$$\frac{P}{\sin(R, W)} = \frac{R}{\sin(W, P)} = \frac{W}{\sin(P, R)},$$

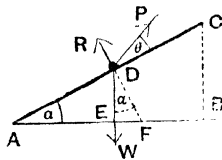
$$\text{i.e., } \frac{P}{\sin(180^\circ - \alpha)} = \frac{R}{\sin(90^\circ + \theta + \alpha)} = \frac{W}{\sin(90^\circ - \theta)},$$

$$\text{i.e., } \frac{P}{\sin \alpha} = \frac{R}{\cos(\theta + \alpha)} = \frac{W}{\cos \theta}.$$

$$\therefore P = W \frac{\sin \alpha}{\cos \theta}, \text{ and } R = W \frac{\cos(\theta + \alpha)}{\cos \theta}.$$

The results of Cases II and III might in a similar manner have been obtained by a direct application of Lami's Theorem.

It will be noted that Case III includes both Cases I and II; if we make θ zero, we obtain Case I; if we put θ equal to $(-\alpha)$, we have Case II.



EXAMPLES. XIX

1. What force, acting horizontally, could keep a mass of 16 lbs. at rest on a smooth inclined plane, whose height is 3 feet and length of base 4 feet, and what is the reaction of the plane?

2. A body rests on an inclined plane, being supported by a force acting up the plane equal to half its weight. Find the inclination of the plane to the horizon and the reaction of the plane.

3. A rope, whose inclination to the vertical is 30° , is just strong enough to support a weight of 180 lbs. on a smooth plane, whose inclination to the horizon is 30° . Find approximately the greatest tension that the rope could exert.

4. A body rests on a plane, inclined at an angle of 60° to the horizon, and is supported by a force inclined at an angle of 30° to the horizon; shew that the force and the reaction of the plane are each equal to the weight of the body.

5. A body rests on a plane, inclined to the horizon at an angle of 30° , being supported by an effort inclined at 30° to the plane; find the ratio of the weight of the body to the effort.

6. A body, of weight $2P$, is kept in equilibrium on an inclined plane by a horizontal force P , together with a force P acting parallel to the plane; find the ratio of the base of the plane to the height and also the reaction of the plane.

7. A body, of 5 lbs. wt., is placed on a smooth plane inclined at 30° to the horizon, and is acted on by two forces, one equal to the weight of 2 lbs. and acting parallel to the plane and upwards, and the other equal to P and acting at an angle of 30° with the plane. Find P and the reaction of the plane.

8. Find the force which acting up an inclined plane will keep a body, of 10 lbs. weight, in equilibrium, it being given that the force, the reaction of the plane, and the weight of the body are in arithmetical progression.

9. A number of loaded trucks, each containing 1 ton, on one part of a tramway inclined at an angle α to the horizon supports an equal number of empty trucks on another part whose inclination is β . Find the weight of a truck.

10. A body, of weight 50 lbs., is in equilibrium on a smooth plane inclined at an angle of $20^\circ 20'$ to the horizon, being supported by a force acting up the plane; find, graphically or by use of trigonometrical tables, the force and the reaction of the plane.

11. A body, of weight 20 lbs., rests on a smooth plane inclined at an angle of 25° to the horizon, being supported by a force P acting at an angle of 35° with the plane; find, graphically or by use of trigonometrical tables, P and the reaction of the plane.

12. A body, of weight 30 lbs., rests on a smooth plane inclined at an angle of $28^\circ 15'$ to the horizon, being supported by a horizontal force P ; find, graphically or by use of trigonometrical tables, P and the reaction of the plane.

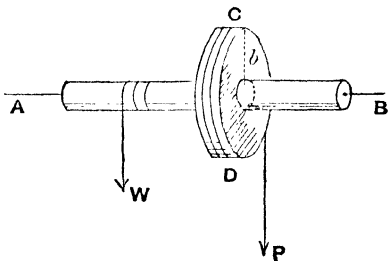
IV. The Wheel and Axle

110. This machine consists of a strong circular cylinder, or axle, terminating in two pivots, A and B , which can turn freely on fixed supports. To the cylinder is rigidly attached a wheel, CD , the plane of the wheel being perpendicular to the axle.

Round the axle is coiled a rope, one end of which is firmly attached to the axle, and the other end of which is attached to the weight.

Round the circumference of the wheel, in a direction opposite to that of the first rope, is coiled a second rope, having one end firmly attached to the wheel, and having the "power" or effort applied at its other end.

The circumference of the wheel is grooved to prevent the rope from slipping off.



111. *To find the relation between the effort and the weight.*

A body, which can turn freely about a fixed axis, is in equilibrium if the algebraic sum of the moments of the forces about the axis vanishes. In this case, the only forces acting on the machine are the effort P and the weight W , which tend to turn the machine in opposite directions. Hence, if a be the radius of the axle, and b be the radius of the wheel, the condition of equilibrium is

$$P \cdot b = W \cdot a.$$

Hence the mechanical advantage = $\frac{W}{P}$

$$= \frac{b}{a} \quad \begin{array}{l} \text{radius of the wheel} \\ \text{radius of the axle} \end{array}.$$

112. Theoretically, by making the quantity $\frac{b}{a}$ very large, we can make the mechanical advantage as great as we please; practically however there are limits. Since the pressure of the fixed supports on the axle must balance P and W , it follows that the thickness of the axle, i.e. $2a$, must not be reduced unduly, for then the axle would break. Neither can the radius of the wheel in practice become very large, for then the machine would be unwieldy. Hence the possible values of the mechanical advantage are bounded, in one direction by the strength of our materials, and in the other direction by the necessity of keeping the size of the machine within reasonable limits.

113. In Art. 111 we have neglected the thicknesses of the ropes. If, however, they are too great to be neglected, compared with the

radii of the wheel and axle, we may take them into consideration by supposing the tensions of the ropes to act along their middle threads.

Suppose the radii of the ropes which pass round the axle and wheel to be x and y respectively; the distances from the line joining the pivots at which the tensions now act are $(a+x)$ and $(b+y)$ respectively. Hence the condition of equilibrium is

$$P(b+y) = W(a+x),$$

so that

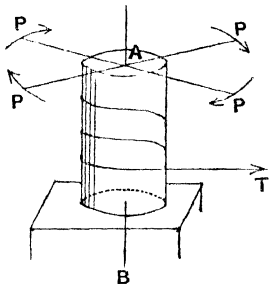
$$\frac{P}{W} = \frac{\text{sum of the radii of the axle and its rope}}{\text{sum of the radii of the wheel and its rope}}.$$

114. Other forms of the Wheel and Axle are the Windlass and Capstan. In these machines the effort instead of being applied, as in Art. 110, by means of a rope passing round a cylinder, is applied at the ends of a spoke, or spokes, which are inserted in a plane perpendicular to the axle.

The Windlass is often used for raising water from a well or for lifting building materials from the ground to a scaffold. The Capstan is employed on ships for lifting anchors.

In the Windlass the axle is horizontal, and in the Capstan it is vertical.

In the latter case the "weight" or resistance consists of the tension T of the rope round the axle, and the effort consists of the forces applied at the ends of bars inserted into sockets at the point A of the axle. The condition of equilibrium may be obtained as in Art. 111.



EXAMPLES. XX

1. If the radii of the wheel and axle be respectively 2 feet and 3 inches, find what effort must be applied to raise a weight of 56 lbs.

2. If the radii of the wheel and axle be respectively 30 inches and 5 inches, find what weight would be supported by a force equal to the weight of 20 lbs., and find also the pressures on the supports on which the axle rests.

If the thickness of the ropes be each 1 inch, find what weight would now be supported.

3. If by means of a wheel and axle an effort equal to 3 lbs. weight balance a weight of 30 lbs., and if the radius of the axle be 2 inches, what is the radius of the wheel?

4. The axle of a capstan is 16 inches in diameter and there are 8 bars. At what distance from the axis must eight men push, one at each bar and each exerting a force equal to the weight of $26\frac{2}{3}$ lbs., in order that they may just produce a strain sufficient to raise the weight of one ton?

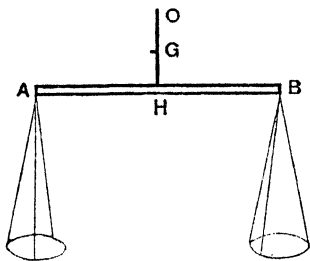
5. Four sailors raise an anchor by means of a capstan, the radius of which is 4 ins. and the length of the spokes 6 feet from the capstan; if each man exert a force equal to the weight of 112 lbs., find the weight of the anchor.

6. Four wheels and axles, in each of which the radii are in the ratio of 5:1, are arranged so that the circumference of each axle is applied to the circumference of the next wheel; what effort is required to support a weight of 1875 lbs.?

V. The Common Balance

115. The Common Balance consists of a rigid beam AB , carrying a scale-pan suspended from each end, which can turn freely about a fulcrum O outside the beam. The fulcrum and the beam are rigidly connected and, if the balance be well constructed, at the point O is a hard steel wedge, whose edge is turned downwards and rests on a small plate of agate.

The body to be weighed is placed in one scale-pan and in the other are placed weights, whose magnitudes are known; these weights are adjusted until the beam of the balance rests in a horizontal position. If OH be perpendicular to the beam, and the arms HA and HB be of equal length, and if the centre of gravity G of the beam lie in the line OH , and the scale-pans be of equal weight, then the weight of the body is the same as the sum of the weights placed in the other scale-pan.



If the weight of the body be not equal to the sum of the weights placed in the other scale-pan, the balance will not rest with its beam horizontal, but will rest with the beam inclined to the horizon.

In the best balances the beam is usually provided with

a long pointer attached to the beam at H . The end of this pointer travels along a graduated scale and, when the beam is horizontal, the pointer is vertical and points to the zero graduation on the scale.

116. *Requisites of a good balance.*

(1) The balance must be **true**.

This will be the case if the arms of the balance be equal, if the weights of the scale-pans be equal, and if the centre of gravity of the beam be on the line through the fulcrum perpendicular to the beam; for the beam will now be horizontal when equal weights are placed in the scale-pans. To test whether the balance is true, first see if the beam is horizontal when the scale-pans are empty; then make the beam horizontal by putting sufficient weights in one scale-pan to balance the weight of a body placed in the other; now interchange the body and the weights; if they still balance one another, the balance must be true; if the beam assumes any position inclined to the vertical, the balance is not true.

(2) The balance should be **sensitive**, *i.e.*, the beam must, for any difference, however small, between the weights in the scale-pans, be inclined at an appreciable angle to the horizon.

(3) The balance should be **stable**, and should quickly take up its position of equilibrium.

It is found that the balance is most sensitive when the distances of the points O and G from the beam AB are very small and that it is most stable when these distances are great.

Hence we see that in any balance great sensitiveness and quick weighing are to a certain extent incompatible. In practice this is not very important; for in balances where great sensitiveness is required (such as balances used in a laboratory) we can afford to sacrifice quickness of weighing; the opposite is the case when the balance is used for ordinary commercial purposes.

To insure as much as possible both the qualities of sensitiveness and quick weighing, the balance should be made with fairly light long arms, and at the same time the distance of the fulcrum from the beam should be considerable.

117. Double Weighing. By this method the weight of a body may be accurately determined even if the balance be not accurate.

Place the body to be weighed in one scale-pan and in the other pan put sand, or other suitable material, sufficient to balance the body. Next remove the body, and in its place put known weights sufficient to again balance the sand. The weight of the body is now clearly equal to the sum of the weights.

This method is used even in the case of extremely good machines when very great accuracy is desired.

118. Ex. 1. *The arms of a balance are equal in length but the beam is unjustly loaded; if a body be placed in each scale-pan in succession and weighed, shew that its true weight is the arithmetic mean between its apparent weights.*

For let the length of the arms be a , and let the horizontal distance of the centre of gravity of the beam from the fulcrum be x .

Let a body, whose true weight is W , appear to weigh W_1 and W_2 successively.

If W' be the weight of the beam, we have

$$W' \cdot a = W' \cdot x + W_1 \cdot a,$$

and

$$W_2 \cdot a = W' \cdot x + W \cdot a. \quad (\text{Art. 59, Cor.})$$

Hence, by subtraction,

$$(W - W_2) a = (W_1 - W) a.$$

$$\therefore W = \frac{1}{2} (W_1 + W_2)$$

= arithmetic mean between the
apparent weights.

Ex. 2. *The arms of a balance are of unequal length, but the beam remains in a horizontal position when the scale-pans are not loaded; shew that, if a body be placed successively in each scale-pan, its true weight is the geometrical mean between its apparent weights.*

Since the beam remains horizontal when there are no weights in the scale-pans, it follows that the centre of gravity of the beam and scale-pans must be vertically under the fulcrum.

Let a and b be the lengths of the arms of the beam and let a body, whose true weight is W , appear to weigh W_1 and W_2 successively.

$$\text{Hence} \quad W \cdot a = W_1 \cdot b \quad \dots\dots\dots(1),$$

$$\text{and} \quad W_2 \cdot a = W \cdot b \quad \dots\dots\dots(2).$$

Hence, by multiplication, we have

$$W^2 \cdot ab = W_1 W_2 \cdot ab.$$

$$\therefore W = \sqrt{W_1 \cdot W_2},$$

i.e., the true weight is the geometrical mean between the apparent weights.

Ex. 3. *If in the previous question the arms be of lengths 11 and 12 inches and if a grocer appear to weigh out 132 lbs. of tea, using alternatively each of the scale-pans, prove that he will defraud himself by half a pound.*

The nominal quantity weighed is 66 lbs. from each scale-pan.

But, by equations (1) and (2) of the previous example, the quantities really weighed are $\frac{11}{12} \cdot 66$ and $\frac{12}{11} \cdot 66$ lbs., i.e. $60\frac{1}{2}$ and 72 lbs., so that altogether he weighs out $132\frac{1}{2}$ lbs. instead of 132 lbs.

EXAMPLES. XXI

1. The only fault in a balance being the unequalness in weight of the scale-pans, what is the real weight of a body which balances 10 lbs. when placed in one scale-pan, and 12 lbs. when placed in the other?

2. The arms of a balance are $8\frac{3}{4}$ and 9 ins. respectively, the goods to be weighed being suspended from the longer arm; find the real weight of goods whose apparent weight is 27 lbs.

3. One scale of a common balance is loaded so that the apparent weight of a body, whose true weight is 18 ounces, is 20 ounces; find the weight with which the scale is loaded.

4. A substance, weighed from the two arms successively of a balance, has apparent weights 9 and 4 lbs. Find the ratio of the lengths of the arms and the true weight of the body.

5. A body, when placed in one scale-pan, appears to weigh 24 lbs. and, when placed in the other, 25 lbs. Find its true weight to three places of decimals, assuming the arms of the scale-pans to be of unequal length.

6. A piece of lead in one pan *A* of a balance is counterpoised by 100 grains in the pan *B*; when the same piece of lead is put into the pan *B* it requires 104 grains in *A* to balance it; what is the ratio of the length of the arms of the balance?

7. A body, placed in a scale-pan, is balanced by 10 lbs. placed in the other pan; when the position of the body and the weights are interchanged, 11 lbs. are required to balance the body. If the length of the shorter arm is 12 ins., find the length of the longer arm and the weight of the body.

8. The arms of a false balance, whose weight is neglected, are in the ratio of 10 : 9. If goods be alternately weighed from each arm, shew that the seller loses $\frac{5}{9}$ ths per cent.

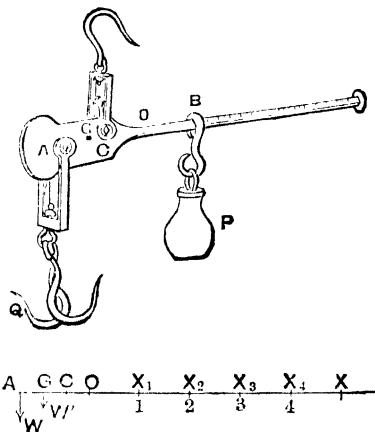
9. If the arms of a false balance be 8 and 9 ins. long respectively, find the prices really paid by a person for tea at two shillings per lb., if the tea be weighed out from the end of (1) the longer, (2) the shorter arm.

10. A dealer has correct weights, but one arm of his balance is $\frac{1}{100}$ th part shorter than the other. If he sell two quantities of a certain drug, each apparently weighing $9\frac{1}{2}$ lbs., at 40s. per lb., weighing one in one scale and the other in the other, what will he gain or lose?

VI. The Steelyards

119. The **Common**, or Roman, Steelyard is a machine for weighing bodies and consists of a rod, AB , movable about a fixed fulcrum at a point C .

At the point A is attached a scale-pan which contains the body to be weighed, and on the arm CB slides a movable weight P . The point at which P must be placed, in order that the beam may rest in a horizontal position, determines the weight of the body in the scale-pan. The arm CB has numbers engraved on it at different points of its length, so that the graduation at which the weight P rests gives the weight of the body.



120. To graduate the Steelyard. Let W' be the weight of the steelyard and the scale-pan, and let G be the point of the beam through which W' acts. The beam is usually constructed so that G lies in the shorter arm AC .

When there is no weight in the scale-pan, let O be the point in CB at which the movable weight P must be placed to balance W' .

Taking moments about C , we have

$$W' \cdot GC = P \cdot CO \dots\dots\dots(i).$$

This condition determines the position of the point O which is the zero of graduation.

When the weight in the scale-pan is W , let X be the point at which P must be placed. Taking moments about C , we have

$$W \cdot CA + W' \cdot GC = P \cdot CX \dots\dots\dots(ii).$$

By subtracting equation (i) from equation (ii), we have

$$W \cdot CA = P \cdot OX.$$

$$\therefore OX = \frac{W}{P} \cdot CA \dots\dots\dots(iii).$$

First, let $W = P$; then, by (iii), we have

$$OX = CA.$$

Hence, if from O we measure off a distance $OX_1 (= CA)$, and if we mark the point X_1 with the figure 1, then, when the movable weight rests here, the body in the scale-pan is P lbs.

Secondly, let $W = 2P$; then, from (iii), $OX = 2CA$.

Hence from O mark off a distance $2CA$, and at the extremity put the figure 2. Thirdly, let $W = 3P$; then, from (iii), $OX = 3CA$, and we therefore mark off a distance from O equal to $3CA$, and at the extremity put the figure 3.

Hence, to graduate the steelyard, we must mark off from O successive distances $CA, 2CA, 3CA, \dots$ and at their extremities put the figures 1, 2, 3, 4, ... The intermediate spaces can be subdivided to shew fractions of P lbs.

If the movable weight be 1 lb., the graduations will shew pounds.

121. The **Danish** steelyard consists of a bar AB , terminating in a heavy knob, or ball, B . At A is attached a scale-pan in which is placed the body to be weighed.

The weight of the body is determined by observing about what point of the bar the machine balances.

[This is often done by having a loop of string, which can slide along the bar, and finding where the loop must be to give equilibrium.]

122. *To graduate the Danish steelyard.* Let P be the weight of the bar and scale-pan, and let G be their common centre of gravity. When a body of weight W is placed in the scale-pan, let C be the position of the fulcrum.

By taking moments about C , we have

$$AC \cdot W = CG \cdot P = (AG - AC) \cdot P.$$

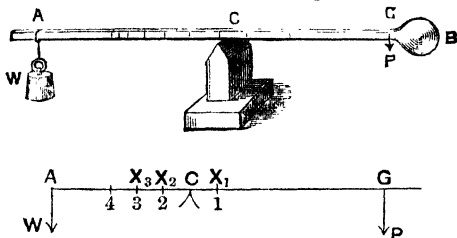
$$\therefore AC(P + W) = P \cdot AG.$$

$$\therefore AC = \frac{P}{P + W} \cdot AG \dots\dots\dots(i).$$

First, let $W = P$; then $AC = \frac{1}{2}AG$.

Hence bisect AG and at the middle point, X_1 , engrave the figure 1; when the steelyard balances about this point the weight of the body in the scale-pan is P .

Secondly, let $W = 2P$; then $AC = \frac{1}{3}AG$.



Take a point X_2 at a distance from A equal to $\frac{1}{3}AG$ and mark it 2.

Next, let W in succession be equal to $3P, 4P, \dots$; from (i), the corresponding values of AC are $\frac{1}{4}AG, \frac{1}{5}AG, \dots$. Take points of the bar at these distances from A and mark them 3, 4, ...

Finally, let $W = \frac{1}{2}P$; then, from (i), $AC = \frac{2}{3}AG$; and let $W = \frac{1}{3}P$; then, from (i), $AC = \frac{3}{4}AG$.

Take points whose distances from A are $\frac{2}{3}AG, \frac{3}{4}AG, \frac{4}{5}AG, \dots$, and mark them $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$.

It will be noticed that the point G can be easily determined; for it is the position of the fulcrum when the steelyard balances without any weight in the scale-pan.

Ex. A Danish steelyard weighs 6 lbs., and the distance of its centre of gravity from the scale-pan is 3 feet; find the distances of the successive points of graduation from the fulcrum.

Taking the notation of the preceding article, we have $P=6$, and $AG=3$ feet.

$$\therefore AC = \frac{6}{6+W} \times 3 = \frac{18}{W+6} \text{ feet.}$$

Hence,

$$\text{when } W=1, AX_1 = \frac{18}{7} = 2\frac{4}{7} \text{ feet,}$$

$$\text{when } W=2, AX_2 = \frac{18}{8} = 2\frac{1}{4} \text{ feet,}$$

$$\text{when } W=3, AX_3 = \frac{18}{9} = 2 \text{ feet,}$$

.....

$$\text{when } W=\frac{1}{2}, AX_{\frac{1}{2}} = \frac{18}{\frac{1}{2}+6} = 2\frac{10}{13} \text{ feet,}$$

and so on.

These give the required graduations.

EXAMPLES. XXII

1. A common steelyard weighs 10 lbs.; the weight is suspended from a point 4 inches from the fulcrum, and the centre of gravity of the steelyard is 3 inches on the other side of the fulcrum; the movable weight is 12 lbs.; where should the graduation corresponding to 1 cwt. be situated?

2. A heavy tapering rod, $14\frac{1}{2}$ inches long and of weight 3 lbs., has its centre of gravity $1\frac{2}{3}$ inches from the thick end and is used as a steelyard with a movable weight of 2 lbs.; where must the fulcrum be placed, so that the steelyard may weigh up to 12 lbs., and what are the intervals between the graduations that denote pounds?

3. In a steelyard, in which the distance of the fulcrum from the point of suspension of the weight is one inch and the movable weight is 6 ozs., to indicate 15 lbs. the weight must be placed 8 inches from the fulcrum; where must it be placed to indicate 24 lbs.?

4. A steelyard, *AB*, 4 feet long, has its centre of gravity 11 inches, and its fulcrum 8 inches, from *A*. If the weight of the machine be 4 lbs. and the movable weight be 3 lbs., find how many inches from *B* is the graduation indicating 15 lbs.

5. A uniform rod, 2 feet long and of weight 3 lbs., is used as a steelyard, whose fulcrum is 2 inches from one end, the sliding weight being 1 lb. Find the greatest and the least weights that can be measured.

Where should the sliding weight be to shew 20 lbs.?

6. In a Danish steelyard the distance between the zero graduation and the end of the instrument is divided into 20 equal parts and the greatest weight that can be weighed is 3 lbs. 9 ozs.; find the weight of the instrument.

7. Find the length of the graduated arm of a Danish steelyard, whose weight is 1 lb., and in which the distance between the graduations denoting 4 and 5 lbs. is one inch.

8. In a Danish steelyard the fulcrum rests halfway between the first and second graduations; shew that the weight in the scale-pan is $\frac{7}{5}$ ths of the weight of the bar.

9. If the weight of a steelyard be worn away to one-half, its length and centre of gravity remaining unaltered, what corrections must be applied to make the weighing true, if the distance of the zero point of graduation from the fulcrum were originally one-third of the distance between successive graduations, and if the movable weight be one pound?

10. A shopman, using a common steelyard, alters the movable weight for which it has been graduated; does he cheat himself or his customers?

CHAPTER X

FRICTION

123. In Art. 18 we defined smooth bodies to be bodies such that, if they be in contact, the only action between them is perpendicular to both surfaces at the point of contact. With smooth bodies, therefore, there is no force tending to prevent one body sliding over the other. If a perfectly smooth body be placed on a perfectly smooth inclined plane, there is no action between the plane and the body to prevent the latter from sliding down the plane, and hence the body will not remain at rest on the plane unless some external force be applied to it.

Practically, however, there are no bodies which are perfectly smooth; there is always *some* force between two bodies in contact to prevent one sliding upon the other. Such a force is called the force of friction.

Friction. Def. *If two bodies be in contact with one another, the property of the two bodies, by virtue of which a force is exerted between the two bodies at their point of contact to prevent one body sliding on the other, is called friction; also the force exerted is called the force of friction.*

124. Friction is a self-adjusting force; no more friction is called into play than is sufficient to prevent motion.

Let a heavy slab of iron with a plane base be placed on a horizontal table. If we attach a piece of string to some point of the body, and pull in a horizontal direction passing through the centre of gravity of the slab, a resistance is felt which prevents our moving the body; this resistance is exactly equal to the force which we exert on the body.

If we now stop pulling, the force of friction also ceases to act; for, if the force of friction did not cease to act, the body would move.

The amount of friction which can be exerted between two bodies is not, however, unlimited. If we continually increase the force which we exert on the slab, we find that finally the friction is not sufficient to overcome this force, and the body moves.

125. Friction plays an important part in the mechanical problems of ordinary life. If there were no friction between our boots and the ground, we should not be able to walk; if there were no friction between a ladder and the ground, the ladder would not rest, unless held, in any position inclined to the vertical; without friction nails and screws would not remain in wood, nor would a locomotive engine be able to draw a train.

126. The laws of statical friction are as follows:

Law I. *When two bodies are in contact, the direction of the friction on one of them at its point of contact is opposite to the direction in which this point of contact would commence to move.*

Law II. *The magnitude of the friction is, when there is equilibrium, just sufficient to prevent the body from moving.*

Suppose, in Art. 109, Case I, the plane to be rough, and that the body, instead of being supported by a force, rested freely on the plane. In this case the force P is replaced by the friction, which is therefore equal to $W \sin \alpha$.

Ex. 1. In what direction does the force of friction act in the case of (1) the wheel of a carriage and (2) the feet of a man who is walking?

Ex. 2. A body, of weight 30 lbs., rests on a rough horizontal plane and is acted upon by a force, equal to 10 lbs. wt., making an angle of 30° with the horizontal; shew that the force of friction is equal to about 8.66 lbs. wt.

Ex. 3. A body, resting on a rough horizontal plane, is acted on by two horizontal forces, equal respectively to 7 and 8 lbs. wt., and acting at an angle of 60° ; shew that the force of friction is equal in magnitude to 13 lbs. wt.

Ex. 4. A body, of weight 40 lbs., rests on a rough plane inclined at 30° to the horizon, and is supported by (1) a force of 14 lbs. wt. acting up the plane, (2) a force of 25 lbs. acting up the plane, (3) a horizontal force equal to 20 lbs. wt., (4) a force equal to 30 lbs. wt. making an angle of 30° with the plane.

Find the force of friction in each case.

Ans. (1) 6 lbs. wt. up the plane; (2) 5 lbs. wt. down the plane; (3) 2.68 lbs. wt. up the plane; (4) 5.98 lbs. wt. down the plane.

127. The above laws hold good, in general; but the amount of friction that can be exerted is limited, and equilibrium is sometimes on the point of being destroyed.

Limiting Friction. Def. *When one body is just on the point of sliding upon another body, the equilibrium is said to be limiting, and the friction then exerted is called limiting friction.*

128. The direction of the limiting friction is given by Law I (Art. 126).

The magnitude of the limiting friction is given by the three following laws:

Law III. *The limiting friction always bears a constant ratio to the normal reaction, and this ratio depends only on the substances of which the bodies are composed.*

Law IV. *The limiting friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.*

Law V. *When motion ensues, by one body sliding over the other, the direction of friction is opposite to the direction of motion; the magnitude of the friction is independent of the velocity, but the ratio of the friction to the normal reaction is slightly less than when the body is at rest and just on the point of motion.*

The above laws are experimental, and cannot be accepted as rigorously accurate; they represent, however, to a fair degree of accuracy the facts under ordinary conditions.

129. Coefficient of Friction. The constant ratio of the limiting friction to the normal pressure is called the coefficient of friction, and is generally denoted by μ ; hence, if F be the friction, and R the normal pressure, between two bodies when equilibrium is on the point of being destroyed, we have $\frac{F}{R} = \mu$, and hence $F = \mu R$.

The values of μ are widely different for different pairs of substances in contact; no pairs of substances are, however, known for which the coefficient of friction is as great as unity.

130. Angle of Friction. When the equilibrium is limiting, if the friction and the normal reaction be compounded into one single force, the angle which this force

makes with the normal is called the angle of friction, and the single force is called the resultant reaction.

Let A be the point of contact of the two bodies, and let AB and AC be the directions of the normal force R and the friction μR .

Let AD be the direction of the resultant reaction S , so that the angle of friction is BAD . Let this angle be λ .

Since R and μR are the components of S , we have

$$S \cos \lambda = R,$$

and $S \sin \lambda = \mu R.$

Hence, by squaring and adding, we have

$$S = R \sqrt{1 + \mu^2},$$

and, by division, $\tan \lambda = \mu.$

Hence we see that *the coefficient of friction is equal to the tangent of the angle of friction.*

131. *If a body be placed upon a rough inclined plane, and be on the point of sliding down the plane under the action of its weight and the reactions of the plane **only**, the angle of inclination of the plane to the horizon is equal to the angle of friction.*

Let θ be the inclination of the plane to the horizon, W the weight of the body, and R the normal reaction.

Since the body is on the point of motion *down* the plane, the friction acts *up* the plane and is equal to μR .

Resolving perpendicular and parallel to the plane, we have

$$W \cos \theta = R,$$

and $W \sin \theta = \mu R.$

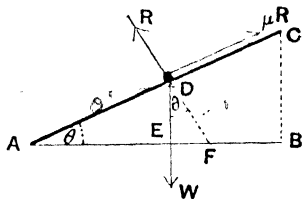
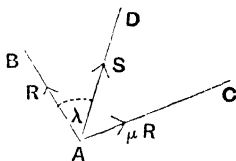
Hence, by division,

$$\tan \theta = \mu = \tan (\text{angle of friction}),$$

$$\therefore \theta = \text{the angle of friction}.$$

This may be shewn otherwise thus:

Since the body is in equilibrium under the action of its weight and



the resultant reaction, the latter must be vertical; but, since the equilibrium is limiting, the resultant reaction makes with the normal the angle of friction.

Hence the angle between the normal and the vertical is the angle of friction, *i.e.*, the inclination EAD of the plane to the horizon, which is equal to the angle EDF , is the angle of friction.

N.B. The student must carefully notice that, when the body rests on the inclined plane *supported by an external force*, it must not be assumed that the coefficient of friction is equal to the tangent of inclination of the plane to the horizon.

132. *To determine the coefficient of friction experimentally.*

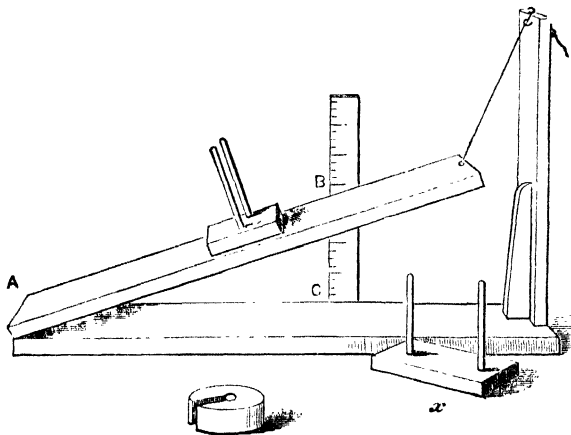
By means of the theorem of the previous article the coefficient of friction between two bodies may be experimentally obtained.

For let an inclined plane be made of one of the substances and let its face be made as smooth as is possible; on this face let there be placed a slab, having a plane face, composed of the other substance.

If the angle of inclination of the plane be gradually increased, until the slab *just* slides, the tangent of the angle of inclination is the coefficient of friction.

To obtain the result as accurately as possible, the experiment should be performed a large number of times with the same substances, and the mean of all the results taken.

In the apparatus here drawn we have a board hinged at one end to another board which can be clamped to a table. The hinged board can be raised or lowered by a string attached to it whose other end passes over the top of a fixed support.



On the hinged board can be placed sliders of different sizes and materials upon which various weights can be placed. Each slider x has two thin brass rods screwed to it on which weights can be piled so that they shall not slip during the experiment. A graduated vertical scale is attached to the lower board, so that the height of the hinged board at B is easily seen. The value of $\frac{BC}{AC}$, i.e., $\tan \theta$ of Art. 131, is then easily obtained.

By this apparatus the laws of friction can be verified; for, within the limits of experiment, it will be found that the value of $\frac{BC}{AC}$, i.e., μ ,

(1) is always the same so long as the slider x is made of the same material in the same state of preparedness of surface,

(2) is independent of the weight put upon the slider, or of its shape,

and (3) is different for different substances.

This method is the one used by Coulomb in the year 1785.

133. Equilibrium on a rough inclined plane.

Ex. 1. A body, of mass 20 lbs., is placed on a rough horizontal plane and is acted on by a force F in a direction making an angle of 60° with the plane; if the coefficient of friction be $\frac{1}{2}$ and the body be on the point of motion, find the value of F and the reaction of the plane.

Let R be the reaction of the plane, so that the friction is $\frac{1}{2}R$.

Since the body is in equilibrium the vertical components of the forces acting on it must balance and so also must the horizontal components (Art. 40).

$$\therefore R + F \sin 60^\circ = 20,$$

and

$$F \cos 60^\circ = \frac{1}{2}R.$$

$$\text{Hence} \quad R + F \frac{\sqrt{3}}{2} = 20, \text{ and } \frac{F}{2} = \frac{1}{2}R.$$

$$\therefore F = R = \frac{40}{2 + \sqrt{3}} = 40(2 - \sqrt{3}) = 40(2 - 1.73205) = 10.718.$$

Hence the force and the reaction are each equal to 10.718 lbs. wt. nearly.

Ex. 2. A body, of mass 30 lbs., is placed on a rough inclined plane whose inclination to the horizon is 60° and is kept in equilibrium by a force acting upwards along the surface of the plane; if the coefficient of friction be $\frac{1}{\sqrt{3}}$, find the magnitude of this force when the body is on the point of sliding (1) up, (2) down, the plane.

Take the figure of Art. 131, and let P be the required force.

(1) When the body is on the point of moving up the plane the friction acts down the plane and is equal to $\frac{1}{\sqrt{3}}R$, where R is the reaction of the plane.

Hence, resolving along and perpendicular to the plane, we have

$$P - \frac{1}{\sqrt{3}} R = 30 \sin 60^\circ = 15\sqrt{3} \dots\dots\dots(i)$$

and

$$R = 30 \cos 60^\circ = 15 \dots\dots\dots(ii).$$

$$\therefore P = \frac{15}{\sqrt{3}} + 15\sqrt{3} = 20\sqrt{3} = 34.64 \text{ lbs. wt. nearly.}$$

(2) When the body is on the point of sliding *down* the plane, the friction acts *up* the plane.

$$\text{In this case we have } P + \frac{1}{\sqrt{3}} R = 30 \sin 60^\circ = 15\sqrt{3},$$

and

$$R = 30 \cos 60^\circ = 15.$$

$$\therefore P = 15\sqrt{3} - 5\sqrt{3} = 10\sqrt{3} = 17.32 \text{ lbs. wt. nearly.}$$

If the force P have any value between the two values thus found the body will be in equilibrium.

EXAMPLES. XXIII

1. A body, of weight 40 lbs., rests on a rough horizontal plane whose coefficient of friction is .25; find the least force which acting horizontally would move the body.

Determine the direction and magnitude of the resultant reaction of the plane.

2. A heavy block with a plane base is resting on a rough horizontal plane. It is acted on by a force at an inclination of 45° to the plane, and the force is gradually increased till the block is just going to slide. If the coefficient of friction be .5, compare the force with the weight of the block.

3. A mass of 30 lbs. is resting on a rough horizontal plane and can be just moved by a force of 10 lbs. wt. acting horizontally; find the coefficient of friction and the direction and magnitude of the resultant reaction of the plane.

4. The height of a rough plane is 5 feet and its length is 13 feet; shew that, if the coefficient of friction be $\frac{1}{3}$, the least force, acting parallel to the plane, that will support 1 cwt. placed on the plane is $8\frac{8}{13}$ lbs. wt.; shew also that the force that would be on the point of moving the body up the plane is $77\frac{7}{13}$ lbs. wt.

5. The base of an inclined plane is 4 feet in length and the height is 3 feet; a force of 8 lbs., acting parallel to the plane, will just prevent a weight of 20 lbs. from sliding down; find the coefficient of friction.

6. A body, of weight 4 lbs., rests in limiting equilibrium on a rough plane whose slope is 30° ; the plane being raised to a slope of 60° , find the force along the plane required to support the body.

7. A weight of 30 lbs. just rests on a rough inclined plane, the height of the plane being $\frac{3}{5}$ ths of its length. Shew that it will require a force of 36 lbs. wt. acting parallel to the plane just to be on the point of moving the weight up the plane.

8. A weight of 60 lbs. is on the point of motion down a rough inclined plane when supported by a force of 24 lbs. wt. acting parallel to the plane, and is on the point of motion up the plane when under the influence of a force of 36 lbs. wt. parallel to the plane; find the coefficient of friction.

9. A cubical block of stone, of weight 5 cwt., is to be drawn along a rough horizontal plane by a force P inclined at 40° to the horizontal. If the angle of friction be 25° , find, by a graphic construction, the least value of P .

10. A body, of weight 1 cwt., rests on a plane inclined at 25° to the horizon, being just prevented from sliding down by a force of 15 lbs. acting up the plane; find, by a graphic construction, the force that will just drag it up and the value of the coefficient of friction.

11. A particle is placed on the outside surface of a rough sphere whose coefficient of friction is μ . Shew that it will be on the point of motion when the radius from it to the centre makes an angle $\tan^{-1} \mu$ with the vertical.

12. How high can a particle rest inside a hollow sphere, of radius a , if the coefficient of friction be $\frac{1}{\sqrt{3}}$?

13. A weight W is kept at rest on a rough plane inclined at 45° to the horizon by means of a horizontal force P . Find the greatest and least possible values of P , the coefficient of friction being < 1 . If the greatest value of P is four times the least value, find the value of μ .

14. Shew that the least force which will move a weight W along a rough horizontal plane is $W \sin \phi$, where ϕ is the angle of friction.

15. At what angle of inclination should the traces be attached to a sledge that it may be drawn up a given hill with the least exertion?

CHAPTER XI

WORK

134. Work. Def. A force is said to do work when its point of application moves in the direction of the force.

The force exerted by a horse, in dragging a waggon, does work.

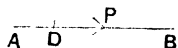
The force exerted by a man, in raising a weight, does work.

The pressure of the steam, in moving the piston of an engine, does work.

When a man winds up a watch or a clock he does work.

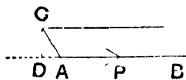
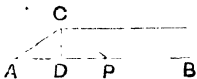
The measure of the work done by a force is the product of the force and the distance through which it moves its point of application in the direction of the force.

Suppose that a force acting at a point A of a body moves the point A to D , then the work done by P is measured by the product of P and AD .



If the point D be on the side of A towards which the force acts, this work is positive; if D lie on the opposite side, the work is negative.

Next, suppose that the point of application of the force is moved to a point C , which does not lie on the line AB . Draw CD perpendicular to AB , or AB produced. Then AD is the distance through which the point of application is moved in the direction of the force. Hence in the first figure the work done is $P \times AD$; in the second figure the work done is $-P \times AD$. When the work done by the force is negative, this is sometimes expressed by saying that the force has work done against it.



In the case when AC is at right angles to AB , the points A and D coincide, and the work done by the force P vanishes.

As an example, if a body be moved about on a horizontal table the work done by its weight is zero. So, again, if a body be moved on an inclined plane, no work is done by the normal reaction of the plane.

135. The unit of work, used in Statics, is called a Foot-Pound, and is the work done by a force, equal to the weight

of a pound, when it moves its point of application through one foot in its own direction. A better, though more clumsy, term than "Foot-Pound" would be Foot-Pound weight.

Thus, the work done by the weight of a body of 10 pounds, whilst the body falls through a distance of 4 feet, is 10×4 foot-pounds.

The work done by the weight of the body, if it were raised through a vertical distance of 4 feet, would be -10×4 foot-pounds.

136. It will be noticed that the definition of work, given in Art. 134, necessarily implies motion. A man may use great exertion in *attempting* to move a body, and yet do no work on the body.

For example, suppose a man pulls at the shafts of a heavily-loaded van, which he cannot move. He may pull to the utmost of his power, but, since the force which he exerts does not move its point of application, he does no work (in the scientific sense of the word).

137. *To find the work done in dragging a body up a smooth inclined plane.*

Taking the figure of Art. 109, Case I, the work done by the force P in dragging the body from A to C is

$$P \times AC.$$

$$\text{But} \quad P = W \sin \alpha.$$

Therefore the work done is $W \sin \alpha \times AC$,

$$\text{i.e., } W \times AC \sin \alpha, \quad \text{i.e., } W \times BC.$$

Hence the work done is the same as that which would be done in lifting the weight of the body through the same height without the intervention of the inclined plane.

138. The previous article is one of the simplest examples of what we shall find to be a universal principle, known as the **Principle of Work**, viz., *Whatever be the machine we use, provided that there be no friction and that the weight of the machine be neglected, the work done by the effort is always equivalent to the work done against the weight, or resistance.*

Assuming that the machine we are using gives mechanical advantage, so that the effort is less than the weight, the distance moved through by the effort is therefore greater than the distance moved through by the weight in the same proportion. This is sometimes expressed in popular language in the form: *What is gained in power is lost in speed.*

A more accurate statement would be that mechanical advantage is always gained at a proportionate diminution of speed. No work is ever gained by the use of a machine though mechanical advantage is generally obtained.

139. In any machine if there be any roughness (as in practice there always is) the work done by the effort is more than the work done against the weight. The principle may be expressed thus,

In any machine, the work done by the effort is equal to the work done against the weight, together with the work done against the frictional resistances of the machine, and the work done against the weights of the component parts of the machine.

As an example, let us take the case of a rough inclined plane, and find the work done in dragging a body up it.

With the figure of Art. 131, let P be the force, acting up the plane, which will be on the point of dragging the body up the plane, so that the friction μR now acts down the plane.

Hence, resolving perpendicular and parallel to the plane, we have

$$W \cos \theta = R,$$

and

$$\mu R + W \sin \theta = P,$$

so that

$$P = W (\sin \theta + \mu \cos \theta).$$

Hence the work done in dragging the body from A to C

$$= P \times AC$$

$$= W (\sin \theta + \mu \cos \theta) \cdot AC$$

$$= W \cdot AC \sin \theta + \mu W \cdot AC \cos \theta$$

$$= W \cdot BC + \mu W \cdot AB$$

= work done in dragging the body through the same vertical height without the intervention of the plane

+ the work done in dragging it along a horizontal distance equal to the base of the inclined plane and of the same roughness as the plane.

The ratio of the work done on the weight to the work done by the effort is, for any machine, called the modulus or efficiency of the machine. We can never get rid entirely of frictional resistances, or make our machine without weight, so that some work must always be lost through these two causes. Hence the efficiency of the machine can never be so great as unity. The more nearly the efficiency approaches to unity, the better is the machine.

There is no machine by whose use we can create work,

and in practice, however smooth and perfect the machine may be, we always lose work. The only use of any machine is to multiply the force we apply, whilst at the same time the distance through which the resistance moves is more than proportionately lessened.

140. The uses of a machine are

(1) to enable a man to lift weights or overcome resistances much greater than he could deal with unaided, *e.g.* by the use of a system of pulleys, or a wheel and axle, or a screw-jacket, etc.,

(2) to cause a motion imparted to one point to be changed into a more rapid motion at some other point, *e.g.* in the case of a bicycle,

or (3) to enable a force to be applied at a more convenient point or in a more convenient manner, *e.g.* in the use of a poker to stir a fire, or in the lifting of a bucket of mortar by means of a rope passing over a pulley at the top of a building, the other end being pulled by a man standing on the ground.

141. Velocity Ratio. The velocity ratio of any machine is the ratio of the distance through which the point of application of the effort or "power" moves to the distance through which the point of application of the resistance, or "weight," moves in the same time; so that

$$\text{Velocity Ratio} = \frac{\text{Distance through which } P \text{ moves}}{\text{Distance through which } W \text{ moves}}$$

If the machine be such that no work has to be done in lifting its component parts, and if it be perfectly smooth throughout, it will be found that the Mechanical Advantage and the Velocity Ratio are equal, so that in this case

$$\frac{W}{P} = \frac{\text{Distance through which } P \text{ moves}}{\text{Distance through which } W \text{ moves}}$$

and then

$$P \times \text{distance through which } P \text{ moves} \\ = W \times \text{distance through which } W \text{ moves,}$$

or, in other words,

$$\text{work done by } P = \text{work done against } W.$$

***142.** The student can verify the truth of the principle, enunciated in Art. 138, for all machines; we shall consider only a few cases.

First system of pulleys (Art. 102).

Neglecting the weights of the pulleys we have, if there be four pulleys,

$$P = \frac{1}{2^4} W.$$

If the weight W be raised through a distance x , the pulley A_2 would, if the distance A_1A_2 remained unchanged, rise a distance x ; but, at the same time, the length of the string joining A_1 to the beam is shortened by x , and a portion x of the string therefore slips round A_1 ; hence, altogether, the pulley A_2 rises through a distance $2x$.

Similarly, the pulley A_3 rises a distance $4x$, and the pulley A_4 a distance $8x$.

Since A_4 rises a distance $8x$, the strings joining it to the beam and to the point at which P is applied both shorten by $8x$.

Hence, since the slack string runs round the pulley A_1 , the point of application of P rises through $16x$.

Hence

$$\frac{\text{work done by the power}}{\text{work done against the weight}} = \frac{P \cdot 16x}{W \cdot x} = \frac{2^4 W \cdot 16x}{W \cdot x} = \frac{W \cdot x}{W \cdot x} = 1.$$

Hence the principle is verified.

Third system of pulleys (Art. 104).

Suppose the weight W to ascend through a space x . The string joining B to the bar shortens by x , and hence the pulley A_3 descends a distance x . Since the pulley A_3 descends x and the bar rises x , the string joining A_3 to the bar shortens by $2x$, and this portion slides over A_3 ; hence the pulley A_2 descends a distance equal to $2x$ together with the distance through which A_3 descends, *i.e.*, A_2 descends a distance $2x + x$, or $3x$. Hence the string A_2F shortens by $4x$, which slips over the pulley A_2 , so that the pulley A_1 descends a distance $4x$ together with the distance through which A_2 descends, *i.e.*, $4x + 3x$, or $7x$. Hence the string A_1G shortens by $8x$, and A_1 itself descends $7x$, so that the point of application of P descends $15x$.

Neglecting the weight of the pulleys, the work done by P therefore

$$= 15x \cdot P = x(2^4 - 1)P = x \cdot W \text{ by equation (1), Art. 104,}$$

$$= \text{work done on the weight } W.$$

Smooth inclined plane (Art. 109, Case III).

Let the body move a distance x along the plane; the distance through which the point of application of P moves, measured along its direction of application, is clearly $x \cos \theta$; also the vertical distance through which the weight moves is $x \sin \alpha$.

Hence the work done by the power is $P \cdot x \cos \theta$, and that done against the weight is $W \cdot x \sin \alpha$. These are equal by the relation proved in Art. 109.

143. Assuming the Principle of Work enunciated in Art. 138 we shall use it to find the conditions of equilibrium of a smooth screw.

A Screw consists of a cylinder of metal round the outside of which runs a protuberant thread of metal.

Let $ABCD$ be a solid cylinder, and let $EFGH$ be a rectangle, whose base EF is equal to the circumference of the solid cylinder. On EH and FG take points

$L, N, Q \dots$ and $K, M, P \dots$

such that

$EL, LN, \dots FK, KM, MP \dots$

are all equal, and join $EK, LM, NP \dots$

Wrap the rectangle round the cylinder, so that the point E coincides with A and EH with the line AD . On being wrapped round the cylinder the point F will coincide with E at A .

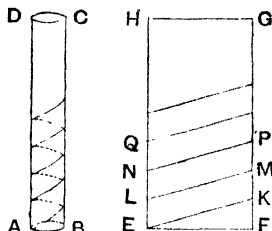
The lines $EK, LM, NP \dots$ will now become a continuous spiral line on the surface of the cylinder and, if we imagine the metal along this spiral line to become protuberant, we shall have the thread of a screw.

It is evident, by the method of construction, that the thread is an inclined plane running round the cylinder and that its inclination to the horizon is the same everywhere. This inclination is often called the angle of the screw, and the distance between two consecutive threads, measured parallel to the axis, is called the pitch of the screw.

The section of the thread of the screw has, in practice, various shapes. The only kind that we shall consider has the section rectangular.

The screw usually works in a fixed support, along the inside of which is cut out a hollow of the same shape as the thread of the screw, and along which the thread slides. The only movement admissible to the screw is to revolve about its axis, and at the same time to move in a direction parallel to its length.

If the screw were placed in an upright position, and a weight placed on its top, the screw would revolve and descend since there is supposed to be no friction between the screw and its support. Hence, if the screw is to remain in equi-



librium, some force must be applied to it; this force is usually applied at one end of a horizontal arm, the other end of which is rigidly attached to the screw.

144. *In a smooth screw, to find the relation between the effort and the weight.*

Let b be the distance, AB , from the axis of the screw, of the point at which the effort P is applied.

Let the arm at the end of which P acts make a complete revolution. The distance through which the point of application of P moves

= circumference of a circle of radius b

= $2\pi b$.

Hence the work done by P is $P \times 2\pi b$.

In the same time the screw rises by a distance equal to that between consecutive threads, i.e. the pitch of screw, so that the work done against the weight is

$W \times$ the pitch of the screw.

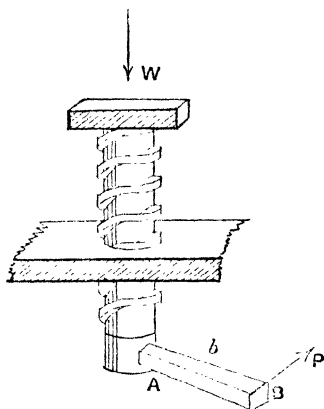
Hence, by the Principle of Work,

$$P \times 2\pi b = W \times \text{pitch of the screw.}$$

Therefore the mechanical advantage = $\frac{W}{P}$

= $\frac{\text{circumference of a circle whose radius is the effort-arm}}{\text{distance between consecutive threads}}$.

Theoretically, the mechanical advantage in the case of the screw can be made as large as we please, by decreasing sufficiently the distance between the threads of the screw. In practice, however, this is impossible; for, if we diminish the distance between the threads to too small a quantity, the threads themselves would not be sufficiently strong to bear the strain put upon them.



EXAMPLES. XXIV

1. Find what mass can be lifted by a smooth vertical screw of $1\frac{1}{2}$ ins. pitch, if the power be a force of 25 lbs. wt. acting at the end of an arm, $3\frac{1}{2}$ feet long.

2. What must be the length of the effort-arm of a screw, having 6 threads to the inch, so that the mechanical efficiency may be 216?

3. What force applied to the end of an arm, 18 ins. long, will produce a pressure of 1100 lbs. wt. upon the head of a screw, when seven turns cause the screw to advance through $\frac{2}{3}$ rds of an inch.

4. A screw, whose pitch is $\frac{1}{4}$ inch, is turned by means of a lever, 4 feet long; find the effort which will raise 15 cwt.

5. The arm of a screw-jack is 1 yard long, and the screw has 2 threads to the inch. What force must be applied to the arm to raise 1 ton?

6. What is the thrust caused by a screw, having 4 threads to the inch, when a force of 50 lbs. wt. is applied to the end of an arm, 2 feet long?

145. Theorem. *To shew that the work done in raising a number of particles from one position to another is Wh , where W is the total weight of the particles, and h is the distance through which the centre of gravity of the particles has been raised.*

Let $w_1, w_2, w_3, \dots w_n$ be the weights of the particles; in the initial position let $x_1, x_2, x_3, \dots x_n$ be their heights above a horizontal plane, and $\bar{x}$ that of their centre of gravity, so that, as in Art. 80, we have

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \dots + w_nx_n}{w_1 + w_2 + \dots + w_n} \dots\dots\dots(1).$$

In the final position let $x'_1, x'_2, \dots x'_n$ be the heights of the different particles, and $\bar{x}'$ the height of the new centre of gravity, so that

$$\bar{x}' = \frac{w_1x'_1 + w_2x'_2 + \dots + w_nx'_n}{w_1 + w_2 + \dots + w_n} \dots\dots\dots(2).$$

But, since $w_1 + w_2 + \dots = W$, equations (1) and (2) give

$$w_1x_1 + w_2x_2 + \dots = W \cdot \bar{x},$$

and

$$w_1x'_1 + w_2x'_2 + \dots = W \cdot \bar{x}'.$$

By subtraction we have

$$w_1(x'_1 - x_1) + w_2(x'_2 - x_2) + \dots = W(\bar{x}' - \bar{x}).$$

But the left-hand member of this equation gives the total work done in raising the different particles of the system from their initial position to their final position; also the right-hand member

$$= W \times \text{height through which the centre of gravity has been raised}$$

$$= W \cdot h.$$

Hence the proposition is proved.

146. Power. Def. *The power of an agent is the amount of work that would be done by the agent if working uniformly for the unit of time.*

The unit of power used by engineers is called a **Horse-Power**. An agent is said to be working with one horse-power when it performs 33,000 foot-pounds in a minute, i.e., when it would raise 33,000 lbs. through a foot in a minute, or when it would raise 330 lbs. through 100 feet in a minute, or 33 lbs. through 1000 feet in a minute.

This estimate of the power of a horse was made by Watt, but is rather above the capacity of ordinary horses. The word Horse-power is usually abbreviated into H. P.

Ex. *A well, of which the section is a square whose side is 4 feet, and whose depth is 300 feet, is full of water; find the work done, in foot-pounds, in pumping the water to the level of the top of the well.*

Find also the H. P. of the engine which would just accomplish this work in one hour.

[N.B. A cubic foot of water weighs 1000 ounces.]

Initially the height of the centre of gravity of the water above the bottom of the well was 150 feet and finally it is 300 feet, so that the height through which the centre of gravity has been raised is 150 feet.

The volume of the water $= 4 \times 4 \times 300$ cubic feet.

Therefore its weight $= 4 \times 4 \times 300 \times \frac{1000}{16}$ lbs. $= 300,000$ lbs.

Hence the work done $= 300,000 \times 150$ ft.-lbs. $= 45,000,000$ ft.-lbs.

Let x be the required H. P. Then the work done by the engine in one hour $= x \times 60 \times 33,000$.

Hence we have $x \times 60 \times 33,000 = 45,000,000$;

$$\therefore x = 22\frac{8}{11}.$$

EXAMPLES. XXV

1. How many cubic feet of water will an engine of 100 H. P. raise in one hour from a depth of 150 feet?

2. A tank, 24 feet long, 12 feet broad, and 16 feet deep, is filled by water from a well the surface of which is always 80 feet below the top

of the tank; find the work done in filling the tank, and the H.P. of an engine, whose modulus is $\cdot 5$, that will fill the tank in 4 hours.

3. A chain, whose mass is 8 lbs. per foot, is wound up from a shaft by the expenditure of four million units of work; find the length of the chain.

4. In how many hours would an engine of 18 H.P. empty a vertical shaft full of water if the diameter of the shaft be 9 feet, and the depth 420 feet?

5. Find the H.P. of an engine that would empty a cylindrical shaft full of water in 32 hours, if the diameter of the shaft be 8 feet and its depth 600 feet.

6. A tower is to be built of brickwork, the base being a rectangle whose external measurements are 22 ft. by 9 ft., the height of the tower 66 feet, and the walls two feet thick; find the number of hours in which an engine of 3 H.P. would raise the bricks from the ground, the weight of a cubic foot of brickwork being 112 lbs.

7. At the bottom of a coal mine, 275 feet deep, there is an iron cage containing coal weighing 14 cwt., the cage itself weighing 4 cwt. 109 lbs., and the wire rope that raises it 6 lbs. per yard. Find the work done when the load has been lifted to the surface, and the H.P. of the engine that can do this work in 40 seconds.

8. In a wheel and axle, if the radius of the wheel be six times that of the axle, and if by means of a power equal to 5 lbs. wt. a body be lifted through 50 feet, find the amount of work expended.

9. A man, whose weight is 10 stone, climbs a rope at the rate of 18 inches per second. Prove that he is working at just under $\frac{2}{5}$ H.P.

10. A steamer is going at the rate of 15 miles per hour; if the effective H.P. of her engines be 10,000, what is the resistance to her motion?

11. A man is cycling at the rate of 6 miles per hour up a hill whose slope is 1 in 20; if the weight of the man and the machine be 200 lbs., prove that he must at the least be working at the rate of $\cdot 16$ H.P.

12. A man rowing 40 strokes per minute propels a boat at the rate of 10 miles an hour, and the resistance to his motion is equal to 8 lbs. wt.; find the work he does in each stroke and the H.P. at which he is working.

13. How much work is done in drawing a load of 6 cwt. up a rough inclined plane, whose height is 3 feet and base 20 feet, the coefficient of friction being $\frac{2}{15}$?

14. A weight of 10 tons is dragged in half an hour through a length of 330 feet up an inclined plane, of inclination 30° , the coefficient of friction being $\frac{1}{\sqrt{3}}$; find the work expended and the H.P. of the engine which could do the work.

DYNAMICS

CHAPTER XII

VELOCITY AND ACCELERATION. RECTILINEAR MOTION

147. A POINT is said to be in motion when it changes its position.

The path of a moving point is the line, straight or curved, which would pass through all the successive positions of the point.

148. If at any instant the position of a moving point be P and at any subsequent instant it be Q , then PQ is its displacement, or change of position, in the intervening time.

To know the displacement of a moving point we must know both the *length* and the *direction* of the line joining the two positions of the point. Hence the displacement of a point involves both magnitude and direction.

149. Velocity. Def. *The velocity of a moving point is the rate of its displacement, i.e., the rate at which it changes its position.*

A velocity therefore possesses both magnitude and direction. We have not fully specified the velocity of a moving point unless we have stated both its rate and its direction of motion.

In Chapters XII—XV we shall consider only the cases in which the *direction* of the velocity of a moving point is constant.

150. A point is said to be moving with uniform velocity when it is moving in a constant direction and passes over equal lengths in equal times, however small these times may be.

Suppose a train described 30 miles in each of three consecutive hours. We are not justified in saying that its velocity is uniform unless we know that it describes half a mile in *each* minute, 44 feet in *each* second, one-millionth of 30 miles in *each* one-millionth of an hour, and so on.

When uniform, the velocity of a moving point is measured by its displacement in each unit of time.

When variable, the velocity is measured at any instant by the displacement which the point would have if it moved during that unit of time with the velocity which it had at the instant under consideration.

By saying that a train is moving with a velocity of 40 miles per hour, we do not mean that it has gone 40 miles in the last hour, or that it will go 40 miles in the next hour, but that, if its velocity remained constant for one hour, then it would describe 40 miles in that hour.

151. The unit of velocity is the velocity of a moving point which has a displacement of a unit of length in each unit of time. In England the units of length and time usually employed are a foot and a second, so that the unit of velocity is the velocity of one foot per second.

A foot is the third part of a yard. A yard is defined to be the distance between the centres of two small gold plugs inserted in a solid brass bar which is kept at Westminster.

In scientific measurements the unit of length usually employed is a centimetre, so that the corresponding unit of velocity is one centimetre per second.

The centimetre is one-hundredth part of a unit which is called a metre and is equal to 39·37 inches approximately, so that a foot = 30·48 centimetres nearly. A decimetre is $\frac{1}{10}$ th and a millimetre $\frac{1}{1000}$ th of a metre.

A metre was meant to be defined as one ten-millionth part of a quadrant of the Earth's surface, *i.e.* of the distance from the North Pole to the Equator. In practice, it is the length of a certain platinum bar kept in Paris.

Since the unit of velocity depends on the units of length and of time, it follows that, if either or both of these be altered, the unit of velocity will also, in general, be altered.

152. If a point be moving with velocity u , then in each unit of time the point moves over u units of length.

Hence in t units of time the point passes over $u \cdot t$ units of length.

Therefore the distance s passed over by a point which moves for time t with velocity u is given by $s = u \cdot t$.

✓ **153. Acceleration. Def.** *The acceleration of a moving point is the rate of change of its velocity.*

The acceleration is uniform when equal changes of velocity

take place in equal intervals of time, however small these times may be.

When uniform, the acceleration is measured by the change in the velocity per unit of time; when variable, it is measured at any instant by what would be the change of the velocity in a unit of time, if during that unit of time the acceleration remained the same as at the instant under consideration.

154. The magnitude of the unit of acceleration is the acceleration of a point which moves so that its velocity is changed by the unit of time.

Hence a point is moving with n units of acceleration when its velocity is changed by n units of velocity in each unit of time.

155. In the simple case of motion in a straight line the only change that the velocity can have is either an increase or a diminution. In the former case the acceleration is positive; in the latter case it is negative and is often called a retardation.

For example suppose a train to be always running due south and that in ten minutes its velocity is uniformly diminished from 30 miles per hour, *i.e.* 44 feet per second, to 15 miles per hour, *i.e.* 22 feet per second. In 600 seconds the loss of velocity is 22 feet per second. Hence in 1 second the loss of velocity is $\frac{22}{600}$ feet per second. This is expressed by saying that its acceleration is $-\frac{22}{600}$ foot-second units.

If in this time the velocity had uniformly increased from 15 miles per hour to 30 miles per hour, the acceleration would have been positive and equal to $\frac{22}{600}$ foot-second units.

156. Theorem. *A point moves in a straight line, starting with velocity u , and moving with constant acceleration f in its direction of motion; if v be its velocity at the end of time t , and s be its distance at that instant from its starting point, then*

$$(1) \quad v = u + ft,$$

$$(2) \quad s = ut + \frac{1}{2}ft^2,$$

and

$$(3) \quad v^2 = u^2 + 2fs.$$

(1) Since f denotes the acceleration, *i.e.* the change in the velocity per unit of time, ft denotes the change in the velocity in t units of time.

But, since the particle possessed u units of velocity initially, at the end of time t it must possess $u + ft$ units of velocity, *i.e.*

$$v = u + ft.$$

✓ (2) Let V be the velocity at the middle of the interval so that, by (1), $V = u + f \cdot \frac{t}{2}$.

Now the velocity changes uniformly throughout the interval t . Hence the velocity at any instant, preceding the middle of the interval by time T , is as much less than V , as the velocity at the same time T after the middle of the interval is greater than V .

Hence, since the time t could be divided into pairs of such equal moments, the space described is the same as if the point moved for time t with velocity V .

$$\therefore s = V \cdot t = \left(u + f \cdot \frac{t}{2}\right) t = ut + \frac{1}{2}ft^2.$$

✓ (3) The third relation can be easily deduced from the first two by eliminating t between them.

$$\begin{aligned} \text{For, from (1),} \quad v^2 &= (u + ft)^2 \\ &= u^2 + 2uft + f^2t^2 \\ &= u^2 + 2f\left(ut + \frac{1}{2}ft^2\right). \end{aligned}$$

$$\text{Hence, by (2),} \quad v^2 = u^2 + 2fs.$$

As an illustration of the method of proof used in this article consider the case of a train which moves on a straight line of rails; let its velocity at 12 noon be 30 feet per second and at 1 p.m. let it be 60 feet per second, and let it have uniformly increased its velocity during the hour, *i.e.* let it have moved with uniform acceleration. At 12.30, the middle of the interval, the velocity was 45 feet per second; at 12.20 and 12.40 the velocity was 40 and 50 feet per second respectively. Clearly 40 is as much less than 45 as 50 is greater than 45. Hence in a short space of time following 12.20 and another equal short space following 12.40, the distance described would be just twice that described in an equal short space of time following 12.30. By reasoning in this manner we see that the total distance described in the hour is the same as what would have been described had the velocity always been what it was at 12.30.

✓ 157. Alternative proof of equation (2).

Let the time t be divided into n equal intervals, each equal to x , so that $t = nx$.

The velocities of the point at the beginnings of these successive intervals are $u, u + fx, u + 2fx, \dots, u + (n-1)fx$.

Hence the space s_1 which *would* be moved through by the point, if it moved during each of these intervals x with the velocity which it has at the *beginning* of each, is

$$\begin{aligned}s_1 &= u \cdot x + (u + fx) \cdot x + (u + 2fx) \cdot x + \dots + [u + (n-1)fx] \cdot x \\ &= n \cdot ux + fx^2 [1 + 2 + 3 + \dots + (n-1)] \\ &= nux + fx^2 \frac{n(n-1)}{2}, \text{ on summing the arithmetic progression,} \\ &= ut + \frac{1}{2}ft^2 \left(1 - \frac{1}{n}\right), \text{ since } x = \frac{t}{n}.\end{aligned}$$

Also the velocities at the ends of these successive intervals are

$$u + fx, u + 2fx, u + 3fx, \dots, u + nfx.$$

Hence the space s_2 which *would* be moved through by the point, if it moved during each of these intervals x with the velocity which it has at the *end* of each, is

$$\begin{aligned}s_2 &= (u + fx) \cdot x + (u + 2fx) \cdot x + (u + 3fx) \cdot x + \dots + (u + nfx) \cdot x \\ &= n \cdot ux + fx^2 [1 + 2 + 3 + \dots + n] \\ &= nux + fx^2 \frac{n(n+1)}{2} \\ &= ut + ft^2 \left(1 + \frac{1}{n}\right), \text{ as before.}\end{aligned}$$

Now the true space s is intermediate between s_1 and s_2 ; also the larger we make n and therefore the smaller the intervals x become, the more nearly do the two hypotheses approach to coincidence.

If we make n infinitely great, and therefore $\frac{1}{n}$ infinitely small, the values of s_1 and s_2 both become $ut + \frac{1}{2}ft^2$.

Hence $s = ut + \frac{1}{2}ft^2$.

158. When the moving point starts from rest we have $u = 0$, and the formulae of Art. 156 take the simpler forms

$$v = ft,$$

$$s = \frac{1}{2}ft^2,$$

and

$$v^2 = 2fs.$$

159. Ex. 1. A point starts with a velocity of 4 feet per second and moves with an acceleration of 1 foot-second unit. What is its velocity at the end of the first minute and how far has it then gone? What is its velocity when it has described 200 feet?

At the end of 60 seconds its velocity, by Art. 156 (1),
 $= 4 + 1 \cdot 60 = 64$ feet per second.

The distance described, by (2),

$$= 4 \cdot 60 + \frac{1}{2} \cdot 1 \cdot 60^2 = 240 + 1800 = 2040 \text{ feet.}$$

Its velocity when it has described 200 feet, by (3),

$$= \sqrt{4^2 + 2 \cdot 1 \cdot 200} = \sqrt{416} = 20.4 \text{ feet per second nearly.}$$

Ex. 2. A train, which is moving at the rate of 60 miles per hour, is brought to rest in 3 minutes with a uniform retardation; find this retardation, and also the distance that the train travels before coming to rest.

$$60 \text{ miles per hour} = \frac{60 \times 1760 \times 3}{60 \times 60} = 88 \text{ feet per second.}$$

Let f be the acceleration with which the train moves.

Since in 180 seconds a velocity of 88 feet per second is destroyed, we have (by formula (1), Art. 156)

$$0 = 88 + 180f.$$

$$\therefore f = -\frac{22}{45} \text{ ft.-sec. units.}$$

[N.B. f has a negative value because it is a retardation.]

Let x be the distance described. By formula (3), we have

$$0 = 88^2 + 2 \left(-\frac{22}{45} \right) \cdot x.$$

$$\therefore x = 88^2 \times \frac{45}{44} = 7920 \text{ feet.}$$

160. Space described in any particular second.

[The student will notice carefully that the formula (2) of Art. 156 gives, not the space traversed in the t th second, but that traversed in t seconds.]

The space described in the t th second

= space described in t seconds – space described in $(t-1)$ seconds

$$\begin{aligned} &= \left[ut + \frac{1}{2}ft^2 \right] - \left[u(t-1) + \frac{1}{2}f(t-1)^2 \right] \\ &= u + \frac{1}{2}f[t^2 - (t-1)^2] \\ &= u + f \frac{2t-1}{2}. \end{aligned}$$

Hence the spaces described in the first, second, third, ... n th seconds of the motion are

$$u + \frac{1}{2}f, u + \frac{3}{2}f, \dots u + \frac{2n-1}{2}f.$$

Ex. A point is moving with uniform acceleration; in the eleventh and fifteenth seconds from the commencement it moves through 24 and 32 feet respectively; find its initial velocity, and the acceleration with which it moves.

Let u be the initial velocity, and f the acceleration.

Then 24 = distance described in the eleventh second

$$= \left[u \cdot 11 + \frac{1}{2}f \cdot 11^2 \right] - \left[u \cdot 10 + \frac{1}{2}f \cdot 10^2 \right].$$

$$\therefore 24 = u + \frac{1}{2}f \dots\dots\dots(1).$$

$$\text{So} \quad 32 = [u \cdot 15 + \frac{1}{2}f \cdot 15^2] - [u \cdot 14 + \frac{1}{2}f \cdot 14^2].$$

$$\therefore 32 = u + \frac{29}{2}f \quad \dots\dots\dots (2).$$

Solving (1) and (2), we have $u = 3$, and $f = 2$.

Hence the point started with a velocity of 3 feet per second, and moved with an acceleration of 2 ft.-sec. units.

EXAMPLES. XXVI

1. The quantities u , f , v , s , and t having the meanings assigned to them in Art. 156,

(1) Given $u = 2$, $f = 3$, $t = 5$, find v and s ;

(2) Given $u = 7$, $f = -1$, $t = 7$, find v and s ;

(3) Given $u = 8$, $v = 3$, $s = 9$, find f and t ;

(4) Given $v = -6$, $s = -9$, $f = -\frac{3}{2}$, find u and t .

The units of length and time are a foot and a second.

2. A body, starting from rest, moves with an acceleration equal to 2 ft.-sec. units; find the velocity at the end of 20 seconds, and the distance described in that time.

3. In what time would a body acquire a velocity of 30 miles per hour, if it started with a velocity of 4 feet per second and moved with the ft.-sec. unit of acceleration?

4. With what uniform acceleration does a body, starting from rest, describe 1000 feet in 10 seconds?

5. A body, starting from rest, moves with an acceleration of 3 centimetre-second units; in what time will it acquire a velocity of 30 centimetres per second, and what distance does it traverse in that time?

6. A point starts with a velocity of 100 cms. per second and moves with -2 centimetre-second units of acceleration. When will its velocity be zero, and how far will it have gone?

7. A body, starting from rest and moving with uniform acceleration, describes 171 feet in the tenth second; find its acceleration.

8. A particle is moving with uniform acceleration; in the eighth and thirteenth seconds after starting it moves through $8\frac{1}{2}$ and $7\frac{1}{2}$ feet respectively; find its initial velocity and its acceleration.

9. In two successive seconds a particle moves through $20\frac{1}{2}$ and $23\frac{1}{2}$ feet respectively; assuming that it was moving with uniform acceleration, find its velocity at the commencement of the first of these two seconds and its acceleration. Find also how far it had moved from rest before the commencement of the first second.

10. A point, moving with uniform acceleration, describes in the last second of its motion $\frac{9}{25}$ ths of the whole distance. If it started from rest, how long was it in motion and through what distance did it move, if it described 6 inches in the first second?

11. A point, moving with uniform acceleration, describes 25 feet in the half second which elapses after the first second of its motion, and 198 feet in the eleventh second of its motion; find the acceleration of the point and its initial velocity.

12. A body moves for 3 seconds with a constant acceleration during which time it describes 81 feet; the acceleration then ceases and during the next 3 seconds it describes 72 feet; find its initial velocity and its acceleration.

13. The speed of a train is reduced from 40 miles per hour to 10 miles per hour whilst it travels a distance of 150 yards; if the retardation be uniform, find how much further it will travel before coming to rest.

14. A point starts from rest and moves with a uniform acceleration of 18 ft.-sec. units; find the time taken by it to traverse the first, second, and third feet respectively.

15. A particle starts from a point O with a uniform velocity of 4 feet per second, and after 2 seconds another particle leaves O in the same direction with a velocity of 5 feet per second and with an acceleration equal to 3 ft.-sec. units. Find when and where it will overtake the first particle.

16. A point moves over 7 feet in the first second during which it is observed, and over 11 and 17 feet in the third and sixth seconds respectively; is this consistent with the supposition that it is subject to a uniform acceleration?

17. A particle starts with a velocity of 200 cms. per second and moves in a straight line with a retardation of 10 cms. per sec. per sec.; find how long elapses before it has described 1500 cms. and explain the double answer.

Motion under Gravity

161. Acceleration of falling bodies. When a heavy body of any kind falls towards the earth, it is a matter of everyday experience that it goes quicker and quicker as it falls, or, in other words, that it moves with an *acceleration*. That it moves with a *constant* acceleration may be shewn by various experiments, one of which will be explained in Art. 192. The first of such experiments was performed by Galileo at Pisa about the year 1590.

From the results of these experiments we learn that, if a body be let fall towards the earth *in vacuo*, it will move with an acceleration which is always the same at the same place on the earth, but which varies slightly for different places.

The value of this acceleration, which is called the

"acceleration due to gravity," is always denoted by the letter " g ."

When foot-second units are used, the value of g varies from about 32.091 at the equator to about 32.252 at the poles. In the latitude of London its value is about 32.19; in other words in the latitude of London the velocity of a body falling *in vacuo* is increased in each second by 32.19 feet per second.

When centimetre-second units are used, the extreme limits are about 978 and 983 respectively, and in the latitude of London the value is about 981.

[*In all numerical examples, unless it is otherwise stated, the motion may be supposed to be in vacuo, and the value of g taken to be 32 when foot-second units, and 981 when centimetre-second units, are used.*]

162. Vertical motion under gravity. Suppose a body is projected vertically from a point on the earth's surface so that it starts with velocity u . The acceleration of the body is opposite to the initial direction of motion, and is therefore denoted by $-g$. Hence the velocity of the body continually gets less and less until it vanishes; the body is then for an instant at rest, but immediately begins to acquire a velocity in a downward direction, and retraces its steps.

163. Time to a given height. The height h at which a body has arrived in time t is given by substituting $-g$ for f in equation (2) of Art. 156, and is therefore given by

$$h = ut - \frac{1}{2}gt^2.$$

This is a quadratic equation with both roots positive; the lesser root gives the time at which the body is at the given height on the way up, and the greater the time at which it is at the same height on the way down.

Thus the time that elapses before a body, which starts with a velocity of 64 feet per second, is at a height of 28 feet is given by

$$28 = 64t - 16t^2, \text{ whence } t = \frac{1}{2} \text{ or } \frac{7}{2}.$$

Hence the particle is at the given height in half a second from the commencement of its motion, and again in 3 seconds afterwards.

164. Velocity at a given height.

The velocity v at a given height h is, by equation (3) of Art. 156, given by $v^2 = u^2 - 2gh$.

Hence the velocity at a given height is independent of the time, and is therefore the same at the same point whether the body be going upwards or downwards.

165. Greatest height attained.

At the highest point the velocity is just zero; hence, if x be the greatest height attained, we have

$$0 = u^2 - 2gx.$$

Hence the greatest height attained $= \frac{u^2}{2g}$.

Also the time T to the greatest height is given by

$$0 = u - gT.$$

$$\therefore T = \frac{u}{g}.$$

166. Velocity due to a given vertical fall from rest.

If a body be dropped from rest, its velocity after falling through a height h is obtained by substituting 0, g , and h for u , f , and s in equation (3) of Art. 156;

$$\therefore v = \sqrt{2gh}.$$

167. Ex. 1. A particle is projected vertically into the air with a velocity of 80 feet per second; find (i) what times elapse before it is at a height of 64 feet, (ii) its velocity when at a height of 40 feet, and (iii) the greatest height it attains.

(i) The required time t is given by

$$64 = 80t - \frac{1}{2}gt^2,$$

$$\text{i.e. by } t^2 - 5t + 4 = 0.$$

$$\therefore t = 1 \text{ or } 4 \text{ seconds.}$$

(ii) The required velocity $= \sqrt{80^2 - 2 \cdot 32 \cdot 40} = \sqrt{6400 - 2560}$

$$= \sqrt{3840} = \text{nearly } 62 \text{ feet per second.}$$

(iii) The greatest height h is given by

$$0 = 80^2 - 2gh.$$

$$\therefore h = \frac{80^2}{64} = 100 \text{ feet.}$$

Ex. 2. A cage in a mine-shaft descends with 2 ft.-sec. units of acceleration. After it has been in motion for 10 seconds a particle is dropped on it from the top of the shaft. What time elapses before the particle hits the cage?

Let T be the time that elapses after the second particle starts. The distance it has fallen through is therefore $\frac{1}{2}gT^2$. The cage has been in

motion for $(T+10)$ seconds, and therefore the distance it has fallen through is

$$\frac{1}{2} \cdot 2 (T+10)^2, \text{ i.e. } (T+10)^2.$$

Hence we have $(T+10)^2 = \frac{1}{2} g T^2 = 16 T^2$.

$$\therefore T+10=4T.$$

$$\therefore T=3\frac{1}{3} \text{ seconds.}$$

Ex. 3. A stone is thrown vertically with the velocity which would just carry it to a height of 100 feet. Two seconds later another stone is projected vertically from the same place with the same velocity; when and where will they meet?

Let u be the initial velocity of projection. Since the greatest height is 100 feet, we have

$$0 = u^2 - 2g \cdot 100.$$

$$\therefore u = \sqrt{2g \cdot 100} = 80.$$

Let T be the time after the first stone starts before the two stones meet.

Then the distance traversed by the first stone in time T

= distance traversed by the second stone in time $(T-2)$.

$$\therefore 80T - \frac{1}{2} g T^2 = 80(T-2) - \frac{1}{2} g (T-2)^2$$

$$= 80T - 160 - \frac{1}{2} g (T^2 - 4T + 4).$$

$$\therefore 160 = \frac{1}{2} g (4T - 4) = 16(4T - 4).$$

$$\therefore T = 3\frac{1}{2} \text{ seconds.}$$

$$\text{Also the height at which they meet} = 80T - \frac{1}{2} g T^2$$

$$= 280 - 196 = 84 \text{ feet.}$$

The first stone will be coming down and the second stone going upwards.

EXAMPLES. XXVII

1. A body is projected from the earth vertically with a velocity of 40 feet per second; find (1) how high it will go before coming to rest, (2) what times will elapse before it is at a height of 9 feet.

2. A particle is projected vertically upwards with a velocity of 40 feet per second. Find (1) when its velocity will be 25 feet per second, and (2) when it will be 25 feet above the point of projection.

3. A stone is thrown vertically upwards with a velocity of 60 feet per second. After what times will its velocity be 20 feet per second, and at what height will it then be?

4. Find (1) the distance fallen from rest by a body in 10 seconds, (2) the time of falling 10 feet, (3) the initial vertical velocity when the body describes 1000 feet downwards in 10 seconds.

5. A stone is thrown vertically into a mine-shaft with a velocity of 96 feet per second, and reaches the bottom in 3 seconds; find the depth of the shaft.

6. A body is projected from the bottom of a mine, whose depth is 88*g* feet, with a velocity of 24*g* feet per second; find the time in which the body, after rising to its greatest height, will return to the surface of the earth again.

7. The greatest height attained by a particle projected vertically upwards is 225 feet; find how soon after projection the particle will be at a height of 176 feet.

8. A body moving in a vertical direction passes a point at a height of 54·5 centimetres with a velocity of 436 centimetres per second; with what initial velocity was it thrown up, and for how much longer will it rise?

9. A particle moving downwards passes a given point with a velocity of fifty metres per second; how long before this was it moving upwards at the same rate?

10. A body is projected vertically upwards with a velocity of 6540 centimetres per second; how high does it rise, and for how long is it moving upwards?

11. Given that a body falling freely passes through 176·99 feet in the sixth second, find the value of *g*.

12. A falling particle in the last second of its fall passes through 224 feet. Find the height from which it fell, and the time of its falling.

13. A body falls freely from the top of a tower, and during the last second of its flight falls $\frac{16}{25}$ ths of the whole distance. Find the height of the tower.

14. Assuming the acceleration of a falling body at the surface of the moon to be one-sixth of its value on the earth's surface, find the height to which a particle will rise if it be projected vertically upwards from the surface of the moon with a velocity of 40 feet per second.

15. A stone *A* is thrown vertically upwards with a velocity of 96 feet per second; find how high it will rise. After 4 seconds from the projection of *A*, another stone *B* is let fall from the same point. Shew that *A* will overtake *B* after 4 seconds more.

16. A body is projected upwards with a certain velocity, and it is found that when in its ascent it is 960 feet from the ground it takes 4 seconds to return to the same point again; find the velocity of projection and the whole height ascended.

17. A body projected vertically downwards described 720 feet in *t* seconds, and 2240 feet in 2*t* seconds; find *t*, and the velocity of projection.

18. A stone is dropped into a well, and the sound of the splash is heard in $7\frac{7}{10}$ seconds; if the velocity of sound be 1120 feet per second, find the depth of the well.

19. A stone is dropped into a well and reaches the bottom with a velocity of 96 feet per second, and the sound of the splash on the water reaches the top of the well in $3\frac{9}{70}$ seconds from the time the stone starts; find the velocity of sound.

20. From a balloon, ascending with a velocity of 32 ft. per second, a stone is let fall and reaches the ground in 17 seconds; how high was the balloon when the stone was dropped?

21. A balloon has been ascending vertically at a uniform rate for 4.5 seconds and a stone let fall from it reaches the ground in 7 seconds. Find the velocity of the balloon and the height from which the stone fell.

22. If a body be let fall from a height of 64 feet at the same instant that another is sent vertically upwards from the foot of the height with a velocity of 64 feet per second, what time elapses before they meet?

If the first body starts 1 second later than the other, what time will elapse?

23. A man ascends the Eiffel Tower to a certain height and drops a stone. He then ascends another 100 feet and drops another stone. The latter takes half a second longer than the former to reach the ground. Neglecting the resistance of the air, find the elevation of the man when he dropped the first stone and the time it took to drop.

24. A particle falling under gravity describes 100 feet in a certain second; how long will it take to describe the next 100 feet, the resistance of the air being neglected?

If the particle, owing to resistance whilst describing the second 100 feet, takes .9 sec., find the ratio of the resistance (assumed to be constant) to the weight of the particle.

25. A particle starts with velocity u and moves with constant acceleration; after moving through 10 feet it acquires a velocity of 20 feet per second, and after moving through 15 feet more its velocity is 30 feet per second. Shew that its velocity will be 40 feet per second after it has described 46 feet from the start, that the time taken is 1.91 seconds approximately, and that $u = 8.165$ feet per second.

CHAPTER XIII

THE LAWS OF MOTION

168. IN the first chapter we stated that the mass of a body was the quantity of matter in the body.

Matter is "that which can be perceived by the senses" or "that which can be acted upon by, or can exert, force."

No definition can however be given that would convey an idea of what matter is to anyone who did not already possess that idea. It, like time and space, is a primary conception.

If we have a small portion of any substance, say iron, resting on a smooth table, we may by a push be able to move it fairly easily; if we take a larger quantity of the same iron, the same effort on our part will be able to move it less easily. Again, if we take two portions of platinum and wood of exactly the same size and shape, the effect produced on these two substances by equal efforts on our part will be quite different. Once more, if we have a croquet-ball and a cannon-ball, both of the same size, lying at rest on the ground, and we kick each of them with the same force, the effect on the first is greater than that on the second. So also we can distinguish between a cask full of water, and an empty one of the same size, by watching the effect of equal kicks applied to them. Thus common experience shews us that the same effort applied to different bodies, under seemingly the same conditions, does not always produce the same result. This is because the *masses* of the bodies are different.

169. The British unit of mass is called the Imperial Pound, and consists of a piece of platinum deposited in the Exchequer Office, of which there are in addition several accurate copies kept in other places of safety.

The French, or scientific, unit of mass is called a gramme, and is the one-thousandth part of a certain quantity of platinum deposited in the Archives. The gramme was meant to be defined as the mass of a cubic centimetre of pure water at a temperature of 4°C .

It is a much smaller unit than a Pound.

One Gramme=about 15·432 grains.

One Pound=about 453·6 grammes.

The system of units in which a centimetre, gramme, and second, are respectively the units of length, mass, and time, is generally called the c.g.s. system of units.

170. The **Momentum**, or Quantity of Motion, of a body is equal to the product of the mass and the velocity of the body. Thus mv is the momentum of a body whose mass is m and which moves with velocity v .

171. We shall now enunciate what are commonly called Newton's Laws of Motion.

"The first two were discovered by Galileo (about the year 1590) and the third in some of its many forms was known to Hooke, Huyghens, Wallis, Wren and others before the publication of the *Principia*." They were put into formal shape by Newton in his *Principia* published in the year 1686.

They are :

Law I. *Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it is compelled by external impressed force to change that state.*

Law II. *The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

Law III. *To every action there is an equal and opposite reaction.*

No formal proof, experimental or otherwise, can be given of these three laws. On them however is based the whole system of Dynamics, and on Dynamics the whole theory of Astronomy. Now the results obtained, and the predictions made, from the theory of Astronomy agree so well with the actual observed facts of Astronomy that it is inconceivable that the fundamental laws on which the subject is based should be erroneous. For example, the Nautical Almanac is published four years beforehand; the motions of the Moon and the Planets are therein predicted and the time and place of Eclipses of the Sun and Moon foretold; and

the predictions in it are always correct. Hence we believe in the truth of the above three laws of motion because the conclusions drawn from them agree with our experience.

172. Law I. We never see this law practically exemplified in nature because it is impossible ever to get rid of all forces during the motion of the body. It may be seen approximately in operation in the case of a piece of dry, hard ice projected along the surface of dry, well swept, ice. The only forces acting on the fragment of ice, in the direction of its motion, are the friction between the two portions of ice and the resistance of the air. The smoother the surface of the ice the further the small portion will go, and the less the resistance of the air the further it will go. The above law asserts that **if** the ice were perfectly smooth and **if** there were no resistance of the air and no other forces acting on the body, then it would go on for ever in a straight line with uniform velocity.

The law states a principle sometimes called the **Principle of Inertia**, viz.—that a body has no *innate* tendency to change its state of rest or of uniform motion in a straight line. If a portion of metal attached to a piece of string is swung round on a smooth horizontal table, then, if the string breaks, the metal, having no longer any force acting on it, proceeds to move in a straight line, viz. the tangent to the circle at the point at which its circular motion ceased.

If a man steps out of a rapidly moving train he is generally thrown to the ground; his feet on touching the ground are brought to rest; but, as no force acts on the upper part of his body, it continues its motion as before, and the man falls to the ground.

If a man is riding on a horse which is galloping at a fairly rapid pace and the horse suddenly stops, the rider is in danger of being thrown over the horse's head.

If a man is seated upon the back seat of a dog-cart, and the latter suddenly starts, the man is very likely to be left behind.

173. Law II. From this law we derive our method of measuring force.

Let m be the mass of a body, and f the acceleration produced in it by the action of a force whose measure is P .

Then, by the second law of motion,

$P \propto$ rate of change of momentum,

$\propto$ rate of change of mv ;

$\propto m \times$ rate of change of v (m being unaltered),

$\propto m \cdot f$.

$\therefore P = \lambda \cdot mf$, where λ is some constant.

Now let the unit of force be so chosen that it may produce in unit mass the unit of acceleration.

Hence, when $m = 1$ and $f = 1$, we have $P = 1$, and therefore

$$\lambda = 1.$$

The unit of force being thus chosen, we have

$$P = m \cdot f.$$

Therefore, when proper units are chosen, the measure of the force is *equal* to the measure of the rate of change of the momentum.

174. From the preceding article it follows that the magnitude of the unit of force used in Dynamics depends on the units of mass, and acceleration, that we use. The unit of acceleration, again, depends, by Arts. 151 and 154, on the units of length and time. Hence the unit of force depends on our units of mass, length, and time. When these latter units are given the unit of force is a determinate quantity.

When a pound, a foot, and a second are respectively the units of mass, length, and time, the corresponding unit of force is called a **Poundal**.

Hence the equation $P = mf$ is a true relation, m being the number of pounds in the body, P the number of poundals in the force acting on it, and f the number of units of acceleration produced in the mass m by the action of the force P on it.

This relation is sometimes expressed in the form

$$\text{Acceleration} = \frac{\text{Moving Force}}{\text{Mass moved}}.$$

N.B. All through this book the unit of force used will be a poundal, unless it is otherwise stated. Thus, when we say that the tension of a string is T , we mean T poundals.

*175. When a gramme, a centimetre, and a second are respectively the units of mass, length, and time, the corresponding unit of force is called a **Dyne**.

Hence when the equation $P=mf$ is used in this system the force must be expressed in dynes, the mass in grammes, and the acceleration in centimetre-second units.

176. Connection between the unit of force and the weight of the unit of mass. As explained in Art. 161, we know that, when a body drops freely *in vacuo*, it moves with an acceleration which we denote by " g "; also the force which causes this acceleration is what we call its weight.

Now the unit of force acting on the unit of mass produces in it the unit of acceleration.

Therefore g units of force acting on the unit of mass produce in it g units of acceleration (by the second law).

But the weight of the unit of mass is that which produces in it g units of acceleration.

Hence the weight of the unit of mass = g units of force.

177. Foot-Pound-Second System of units. In this system g is equal to 32.2 approximately.

Therefore the weight of one pound is equal to g units of force, *i.e.* to g poundals, where $g = 32.2$ approximately.

Hence a poundal is approximately equal to $\frac{1}{32.2}$ times the weight of a pound, *i.e.* to about the weight of half an ounce.

Since g has different values at different points of the earth's surface, and since a poundal is a force which is the same everywhere, it follows that **the weight of a pound is not constant, but has different values at different points of the earth's surface.**

*178. *Centimetre-Gramme-Second System of units.* In this system g is equal to 981 approximately.

Therefore the weight of one gramme is equal to g units of force, *i.e.* to g dynes, where

$$g = 981 \text{ approximately.}$$

Hence a dyne is equal to the weight of about $\frac{1}{981}$ of a gramme.

The dyne is a much smaller unit than a poundal. The approximate relation between them may be easily found as follows:

$$\begin{aligned}\frac{\text{One Poundal}}{\text{One Dyne}} &= \frac{\frac{1}{32.2} \text{ wt. of a pound}}{\frac{1}{981} \text{ wt. of a gramme}} \\ &= \frac{981}{32.2} \times \frac{\text{one pound}}{\text{one gramme}} = \frac{981}{32.2} \times 453.6 \text{ (by Art. 169).}\end{aligned}$$

Hence One Poundal = about 13,800 dynes.

Ex. 1. A mass of 20 pounds is acted on by a constant force which in 5 seconds produces a velocity of 15 feet per second. Find the force, if the mass was initially at rest.

From the equation $v = u + ft$, we have $f = \frac{15}{5} = 3$.

Also, if P be the force expressed in poundals, we have

$$P = 20 \times 3 = 60 \text{ poundals.}$$

Hence P is equal to the weight of about $\frac{60}{32}$, i.e. $1\frac{7}{8}$, pounds.

Ex. 2. A mass of 10 pounds is placed on a smooth horizontal plane, and is acted on by a force equal to the weight of 3 pounds; find the distance described by it in 10 seconds.

Here the moving force = weight of 3 lbs. = $3g$ poundals;

and the mass moved = 10 poundals.

Hence, using ft.-sec. units, the acceleration = $\frac{3g}{10}$,

so that the distance required = $\frac{1}{2} \cdot \frac{3g}{10} \cdot 10^2 = 480$ feet.

Ex. 3. Find the magnitude of the force which, acting on a kilogramme for 5 seconds, produces in it a velocity of one metre per second.

Here the velocity acquired = 100 cms. per sec.

Hence the acceleration = 20 c.g.s. units.

Hence the force = 1000×20 dynes = weight of about $\frac{1000 \times 20}{981}$ or 20.4 grammes.

EXAMPLES. XXVIII

1. Find the acceleration produced when

(1) A force of 5 poundals acts on a mass of 10 pounds.

(2) A force equal to the weight of 5 pounds acts on a mass of 10 pounds.

(3) A force of 50 pounds weight acts on a mass of 10 tons.

2. Find the force expressed (1) in poundals, (2) in terms of the weight of a pound, that will produce in a mass of 20 pounds an acceleration of 10 foot-second units.

3. Find the force which, acting horizontally for 5 seconds on a mass of 160 pounds placed on a smooth table, will generate in it a velocity of 15 feet per second.

4. Find the magnitude of the force which, acting on a mass of 10 cwt. for 10 seconds, will generate in it a velocity of 3 miles per hour.

5. A force, equal to the weight of 2 lbs., acts on a mass of 40 lbs. for half a minute; find the velocity acquired, and the space moved through, in this time.

6. A body, acted upon by a uniform force, in ten seconds describes a distance of 25 feet; compare the force with the weight of the body, and find the velocity acquired.

7. In what time will a force, which is equal to the weight of a pound, move a mass of 18 lbs. through 50 feet along a smooth horizontal plane, and what will be the velocity acquired by the mass?

8. A body, of mass 200 tons, is acted on by a force equal to 112,000 poundals; how long will it take to acquire a velocity of 30 miles per hour?

9. In what time will a force, equal to the weight of 10 lbs., acting on a mass of 1 ton move it through 14 feet?

10. A mass of 224 lbs. is placed on a smooth horizontal plane, and a uniform force acting on it parallel to the table for 5 seconds causes it to describe 50 feet in that time; shew that the force is equal to about 28 lbs. weight.

11. A heavy truck, of mass 16 tons, is standing at rest on a smooth line of rails. A horse now pulls at it steadily in the direction of the line of rails with a force equal to the weight of 1 cwt. How far will it move in 1 minute?

12. A 30-ton mass is moving on smooth horizontal rails at the rate of 20 miles per hour; what force would stop it in (1) half a minute, and (2) in half a mile?

13. A force equal to the weight of 10 grammes acts on a mass of 27 grammes for 1 second; find the velocity of the mass and the distance it has travelled over. At the end of the first second the force ceases to act; how far will the body travel in the next minute?

14. A force equal to the weight of a kilogramme acts on a body continuously for 10 seconds, and causes it to describe 10 metres in that time; find the mass of the body.

15. A mass which starts from rest is acted upon by a force which in $\frac{1}{200}$ th of a second communicates to it a velocity of 3 miles per hour; find the ratio of the force to the weight of the mass.

16. A horizontal force equal to the weight of 9 lbs. acts on a mass along a smooth horizontal plane; after moving through a space of 25 feet the mass has acquired a velocity of 10 feet per second; find its magnitude.

17. A body is placed on a smooth table and a force equal to the weight of 6 lbs. acts continuously on it; at the end of 3 seconds the body is moving at the rate of 48 feet per second; find its mass.

18. A body, of mass 3 lbs., is falling under gravity at the rate of 100 feet per second. What is the uniform force that will stop it (1) in 2 seconds, (2) in 2 feet?

19. Of two forces, one acts on a mass of 5 lbs. and in one-eleventh of a second produces in it a velocity of 5 feet per second, and the other acting on a mass of 625 lbs. in 1 minute produces in it a velocity of 18 miles per hour; compare the two forces.

20. A mass of 10 lbs. falls 10 feet from rest, and is then brought to rest by penetrating 1 foot into some sand; find the average force exerted by the sand on it.

21. A bullet moving at the rate of 200 feet per second is fired into a trunk of wood into which it penetrates 9 inches; if a bullet moving with the same velocity were fired into a similar piece of wood 5 inches thick, with what velocity would it emerge, supposing the resistance to be uniform?

22. A cannon-ball of mass 1000 grammes is discharged with a velocity of 45,000 centimetres per second from a cannon the length of whose barrel is 200 centimetres; shew that the mean force exerted on the ball during the explosion is 5.0625×10^9 dynes.

23. A motor-car travelling at the rate of 40 kilometres per hour is stopped by its brakes in 4 seconds; shew that it will go about 22 metres from the point at which the brakes are first applied, and that the force exerted by them is about .283 times the weight of the car, and would hold the car at rest on an incline of about 1 in $3\frac{1}{2}$.

179. A poundal and a dyne are called **Absolute Units** because their values are not dependent on the value of g , which varies at different places on the earth's surface. The *weight* of a pound and of a gramme do depend on this value. Hence they are called **Gravitation Units**.

180. *The weight of a body is proportional to its mass and is independent of the kind of matter of which it is composed.* The following is an experimental fact: If we have an airtight receiver, and if we allow to drop at the same instant, from the same height, portions of matter of any kind whatever, such as a piece of metal, a feather, a piece of paper etc., all these substances will be found to have always fallen through the same distance, and to hit the base of the receiver at the same time, whatever be the substances, or the height

from which they are allowed to fall. Since these bodies always fall through the same height in the same time, therefore their velocities [rates of change of space,] and their accelerations [rates of change of velocity,] must be always the same.

The student can approximately perform the above experiment without creating a vacuum. Take a penny and a light substance, say a small piece of paper; place the paper on the penny, held horizontally, and allow both to drop. They will be found to keep together in their fall, although, if they be dropped separately, the penny will reach the ground much quicker than the paper. The penny clears the air out of the way of the paper, and so the same result is produced as would be the case if there were no air.

Let W_1 and W_2 poundals be the weights of any two of these bodies, m_1 and m_2 their masses. Then since their accelerations are the same and equal to g , we have

$$W_1 = m_1 g,$$

and

$$W_2 = m_2 g;$$

$$\therefore W_1 : W_2 :: m_1 : m_2,$$

or the weight of a body is proportional to its mass.

Hence bodies whose weights are equal have equal masses; so also the ratio of the masses of two bodies is known when the ratio of their weights is known.

N.B. The equation $W = mg$ is a numerical one, and means that the number of units of force in the weight of a body is equal to the product of the number of units of mass in the mass of the body, and the number of units of acceleration produced in the body by its weight.

181. Distinction between mass and weight. The student must carefully notice the difference between the mass and the weight of a body. He has probably been so accustomed to estimate the masses of bodies by means of their weights that he has not clearly distinguished between the two. If it were possible to have a cannon-ball at the centre of the earth it would have no weight there; for the attraction of the earth on a particle at its centre is zero. If, however, it were in motion, the same force would be required to stop it as would be necessary under similar conditions at the surface of the earth. Hence we see that it might be possible for a body to have no weight; its mass however remains unaltered.

The confusion is probably to a great extent caused by the fact that the word "pound" is used in two senses which are scientifically different; it is used to denote both what we more properly call "the

mass of one pound" and "the weight of one pound." It cannot be too strongly impressed on the student that, strictly speaking, a pound is a mass and a mass only; when we wish to speak of the force with which the earth attracts this mass we ought to speak of the "weight of a pound." This latter phrase is often shortened into "a pound," but care must be taken to see in which sense this word is used.

It may also be noted here that the expression "a ball of lead weighing 20 lbs." is, strictly speaking, an abbreviation for "a ball of lead whose weight is equal to the weight of 20 lbs." The mass of the lead is 20 lbs.; its weight is 20*g* poundals.

182. Weighing by Scales and a Spring Balance. We have pointed out (Art. 161) that the acceleration due to gravity, *i.e.* the value of *g*, varies slightly as we proceed from point to point of the earth's surface. When we weigh out a substance (say tea) by means of a pair of scales, we adjust the tea until the weight of the tea is the same as the weight of sundry pieces of metal whose masses are known, and then, by Art. 180, we know that the mass of the tea is the same as the mass of the metal. Hence a pair of scales really measures masses and not weights, and so the apparent weight of the tea is the same everywhere.

When we use a spring balance, we compare the *weight* of the tea with the *force* necessary to keep the spring stretched through a certain distance. If then we move our tea and spring balance to another place, say from London to Paris, the weight of the tea will be different, whilst the force necessary to keep the spring stretched through the same distance as before will be the same. Hence the weight of the tea will pull the spring through a distance different from the former distance, and hence its apparent weight as shewn by the instrument will be different.

If we have two places, *A* and *B*, at the first of which the numerical value of *g* is greater than at the second, then a given mass of tea will [as tested by the spring balance] appear to weigh more at *A* than it does at *B*.

Ex. 1. At the equator the value of *g* is 32.09 and in London the value is 32.2; a merchant buys tea at the equator, at a shilling per pound, and sells in London; at what price per pound (apparent) must he sell so that he may neither gain nor lose, if he use the same spring balance for both transactions?

A quantity of tea which weighs 1 lb. at the equator will appear to weigh $\frac{32.2}{32.09}$ lb. in London. Hence he should sell $\frac{32.2}{32.09}$ lb. for one shilling, or at the rate of $\frac{3209}{3220}$ shilling per pound.

Ex. 2. At a place *A*, *g* = 32.24, and at a place *B*, *g* = 32.12. A merchant buys goods at £10 per cwt. at *A* and sells at *B*, using the same spring balance. If he is to neither gain nor lose, shew that his selling price must be £10. 0*s.* 9*d.* per cwt. nearly.

183. Law III. *To every action there is an equal and opposite reaction.*

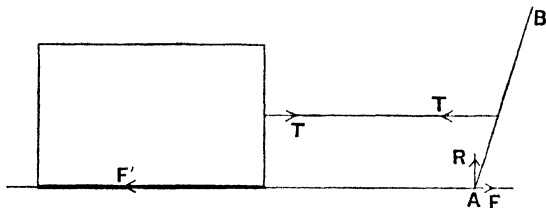
Every exertion of force consists of a mutual action between two bodies. This mutual action is called the stress between the two bodies, so that the Action and Reaction of Newton together form the Stress.

Illustrations. 1. If a book rests on a table, the book presses the table with a force equal and opposite to that which the table exerts on the book.

2. If a man raises a weight by means of a string tied to it, the string exerts on the man's hand exactly the same force that it exerts on the weight, but in the opposite direction.

3. The attraction of the earth on a body is its weight, and the body attracts the earth with a force equal and opposite to its own weight.

4. When a man drags a heavy body along the ground by means of a rope, the rope drags the man back with a force equal to that with which it drags the body forward. [The weight of the rope is neglected.]



[In the figure AB represents the central line of the man's body. F and R are the horizontal and vertical forces which the earth exerts on his feet, and which are equal and opposite to the forces his feet exert on the earth. T is the tension of the rope which acts in opposite directions at its ends. F' is the horizontal force between the earth and the body.]

The man moves because $F > T$.

The body moves because $T > F'$.

Thus at the commencement of the motion we have $F > T > F'$.

When the man and body are moving uniformly these three forces are equal.]

5. In the case of a stretched piece of indiarubber, with the ends held in a man's hands, the indiarubber pulls one hand with a force equal and opposite to that with which it pulls the other hand.

6. The compressed buffers between two railway carriages push one carriage with a force exactly equal and opposite to that with which they push the other carriage.

CHAPTER XIV

THE LAWS OF MOTION (*continued*). APPLICATION TO SIMPLE PROBLEMS

184. Motion of two particles connected by a string.

Two particles, of masses m_1 and m_2 , are connected by a light inextensible string which passes over a small smooth pulley. If m_1 be $> m_2$, find the resulting motion of the system, and the tension of the string.

Let the tension of the string be T poundals; the pulley being smooth, this will be the same throughout the string.

Since the string is inextensible, the velocity of m_2 upwards must, throughout the motion, be the same as that of m_1 downwards.

Hence their accelerations [rates of change of velocity] must be the same in magnitude. Let the magnitude of the common acceleration be f .

Now the force on m_1 downwards is $m_1g - T$ poundals.

Hence $m_1g - T = m_1f$ (1).

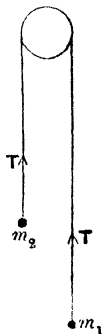
So the force on m_2 upwards is $T - m_2g$ poundals;

$$\therefore T - m_2g = m_2f$$
(2).

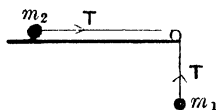
Adding (1) and (2), we have $f = \frac{m_1 - m_2}{m_1 + m_2}g$, giving the common acceleration.

Also, from (2), $T = m_2(f + g) = \frac{2m_1m_2}{m_1 + m_2}g$ poundals.

Since the acceleration is known and constant, the equations of Art. 156 give the space moved through and the velocity acquired in any given time.



185. Two particles, of masses m_1 and m_2 , are connected by a light inextensible string; m_2 is placed on a smooth horizontal table and the string passes over the edge of the table, m_1 hanging freely; find the resulting motion.



Let the tension of the string be T poundals.

The velocity and acceleration of m_2 along the table must be equal to the velocity and acceleration of m_1 in a vertical direction.

Let f be the common acceleration of the masses.

The force on m_1 downwards is

$$m_1 g - T;$$

$$\therefore m_1 g - T = m_1 f \dots\dots\dots(1).$$

The only horizontal force acting on m_2 is the tension T [for the weight of m_2 is balanced by the reaction of the table].

$$\therefore T = m_2 f \dots\dots\dots(2).$$

Adding (1) and (2), we have

$$m_1 g = (m_1 + m_2) f.$$

$$\therefore f = \frac{m_1}{m_1 + m_2} g,$$

giving the required acceleration.

Hence, from (2), $T = \frac{m_1 m_2}{m_1 + m_2} g$ poundals = weight of a body whose mass is

$$\frac{m_1 m_2}{m_1 + m_2}.$$

186. Ex. Two particles, of masses 11 and 13 lbs., are connected by a light string passing over a small smooth pulley. Find (1) the velocity at the end of 4 seconds, and (2) the space described in 4 seconds. If at the end of 4 seconds the string be cut, find the distance described by each in the next 2 seconds.

If T poundals be the tension of the string and f the common acceleration, we have

$$13g - T = 13f,$$

and

$$T - 11g = 11f.$$

Hence, by addition $f = \frac{g}{12}$.

The common velocity at the end of 4 seconds $= 4f = \frac{4 \times 32}{12} = 10\frac{2}{3}$ feet per second.

$$\begin{aligned}\text{The space described in this time} &= \frac{1}{2} \cdot f \cdot 4^2 \\ &= 8 \times \frac{32}{12} = 21\frac{1}{3} \text{ feet.}\end{aligned}$$

If the string be now cut the larger mass starts downward with velocity $10\frac{2}{3}$ and acceleration g ; the smaller starts upward with velocity $10\frac{2}{3}$ and with an acceleration $-g$.

$$\begin{aligned}\text{The space described by the larger mass in the next 2 seconds} \\ = 10\frac{2}{3} \times 2 + \frac{1}{2} \cdot g \cdot 2^2 = 21\frac{1}{3} + 64 = 85\frac{1}{3} \text{ feet.}\end{aligned}$$

$$\begin{aligned}\text{The space described by the smaller mass} \\ = 10\frac{2}{3} \times 2 - \frac{1}{2} \cdot g \cdot 2^2 = 21\frac{1}{3} - 64 = -42\frac{2}{3} \text{ feet.}\end{aligned}$$

In these two seconds the upward velocity of the smaller mass has been destroyed, and it has fallen to a point which is $42\frac{2}{3}$ feet below the point at which it was when the string was cut.

EXAMPLES. XXIX

1. A mass of 9 lbs., descending vertically, drags up a mass of 6 lbs. by means of a string passing over a smooth pulley; find the acceleration of the system and the tension of the string.

2. Two particles, of masses 7 and 9 lbs., are connected by a light string passing over a smooth pulley. Find (1) their common acceleration, (2) the tension of the string, (3) the velocity at the end of 5 seconds, and (4) the distance described in 5 seconds.

3. Masses of 14 and 18 ounces are connected by a thread passing over a light pulley; how far do they go in the first 3 seconds of the motion, and what is the tension of the string?

4. To the two ends of a light string passing over a small smooth pulley are attached masses of 977 grammes and x grammes; find x so that the former mass may rise through 200 centimetres in 10 seconds.

5. Two masses of 50 and 70 grammes are fastened to the ends of a cord passing over a frictionless pulley supported by a hook. When they are free to move, shew that the pull on the hook is equal to $116\frac{2}{3}$ grammes weight.

6. Two equal masses, of 3 lbs. each, are connected by a light string hanging over a smooth peg; if a third mass of 3 lbs. be laid on one of them, by how much is the pressure on the peg increased?

7. Two masses, each equal to m , are connected by a string passing over a smooth pulley; what mass must be taken from one and added to the other, so that the system may describe 200 feet in 5 seconds?

8. A mass of 3 lbs., descending vertically, draws up a mass of 2 lbs. by means of a light string passing over a pulley; at the end of 5 seconds the string breaks; find how much higher the 2 lb. mass will go.

9. A body, of mass 9 lbs., is placed on a smooth table at a distance of 8 feet from its edge, and is connected, by a string passing over the edge, with a body of mass 1 lb.; find

- (1) the common acceleration,
- (2) the time that elapses before the body reaches the edge of the table,
- and (3) its velocity on leaving the table.

10. A mass of 19 ounces is placed on a smooth table and connected by a light string passing over the edge of the table with a mass of 5 ounces which hangs vertically; find the acceleration of the masses and the tension of the string.

11. A mass of 70 lbs. is placed on a smooth table at a distance of 8 feet from its edge and connected by a light string passing over the edge with a mass of 10 lbs. hanging freely; what time will elapse before the first mass will leave the table?

12. A mass of 100 grammes is attached by a string passing over a smooth pulley to a larger mass; find the magnitude of the latter so that, if after the motion has continued 3 seconds the string be cut, the former will ascend 54·5 centimetres before descending.

13. Two scale-pans, of mass 3 lbs. each, are connected by a string passing over a smooth pulley; shew how to divide a mass of 12 lbs. between the two scale-pans so that the heavier may descend a distance of 50 feet in the first 5 seconds.

14. Two strings pass over a smooth pulley; on one side they are attached to masses of 3 and 4 lbs. respectively, and on the other to one of 5 lbs.; find the tensions of the strings and the acceleration of the system.

15. A string hung over a pulley has at one end a weight of 10 lbs. and at the other end weights of 8 and 4 lbs. respectively; after being in motion for 5 seconds the 4 lb. weight is taken off; find how much further the weights go before they first come to rest.

187. Motion down a smooth inclined plane.

Let α be the inclination of the plane to the horizon. If a particle be sliding down the plane the only forces acting on it are its weight mg vertically downwards and the normal reaction of the plane.

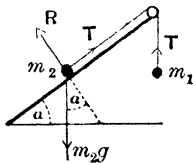
The weight mg may be resolved (as in Art. 27) into $mg \sin \alpha$ down the plane and $mg \cos \alpha$ perpendicular to the plane. The latter force is balanced by the normal reaction and the former produces the acceleration f down the plane.

Hence $mg \sin \alpha = mf$.

The acceleration of the particle down the plane is therefore $g \sin \alpha$.

The velocity of the particle after it has described a length l of the plane $= \sqrt{2g \sin \alpha \cdot l} = \sqrt{2gh}$, where h is the height of the plane, and is therefore the same as that of a particle which has dropped vertically through a distance equal to the height of the plane.

188. Two masses, m_1 and m_2 , are connected by a string; m_2 is placed on a smooth plane inclined at an angle α to the horizon, and the string, after passing over a small smooth pulley at the top of the plane, is attached to m_1 , which hangs vertically; if m_1 descends, find the resulting motion.



Let the tension of the string be T' poundals. The velocity and acceleration of m_2 up the plane are clearly equal to the velocity and acceleration of m_1 vertically.

Let f be this common acceleration. For the motion of m_1 , we have

$$m_1 g - T = m_1 f \dots\dots\dots (1).$$

The weight of m_2 is $m_2 g$ vertically downwards.

The resolved part of $m_2 g$ perpendicular to the inclined plane is balanced by the reaction R of the plane, since m_2 has no acceleration perpendicular to the plane.

The resolved part of the weight down the inclined plane is $m_2 g \sin \alpha$, and hence the total force up the plane is

$$T - m_2 g \sin \alpha.$$

Hence $T - m_2 g \sin \alpha = m_2 f \dots\dots\dots (2).$

Adding (1) and (2), we easily have

$$f = \frac{m_1 - m_2 \sin \alpha}{m_1 + m_2} g.$$

Also, substituting in (1),

$$\begin{aligned} T &= m_1 (g - f) = m_1 g \left[1 - \frac{m_1 - m_2 \sin \alpha}{m_1 + m_2} \right] \\ &= \frac{m_1 m_2 (1 + \sin \alpha)}{m_1 + m_2} g \text{ poundals,} \end{aligned}$$

giving the tension of the string.

189. Motion on a rough plane. *A particle slides down a rough plane inclined to the horizon at an angle α ; if μ be the coefficient of friction, to determine the motion.*

Let m be the mass of the particle, so that its weight is mg pounds; let R be the reaction of the plane, and μR the friction.

The total force perpendicular to the plane is $(R - mg \cos \alpha)$ pounds.

The total force down the plane is

$$(mg \sin \alpha - \mu R)$$

pounds.

Now perpendicular to the plane there cannot be any motion, and hence there is no change of motion.

Hence the acceleration, and therefore the total force, in that direction is zero.

$$\therefore R - mg \cos \alpha = 0 \dots \dots \dots (1).$$

Also the acceleration down the plane

$$= \frac{\text{moving force}}{\text{mass moved}} = \frac{mg \sin \alpha - \mu R}{m} = g(\sin \alpha - \mu \cos \alpha), \text{ by (1).}$$

Hence the velocity of the particle after it has moved from rest over a length l of the plane is, by Art. 158, equal to

$$\sqrt{2gl(\sin \alpha - \mu \cos \alpha)}.$$

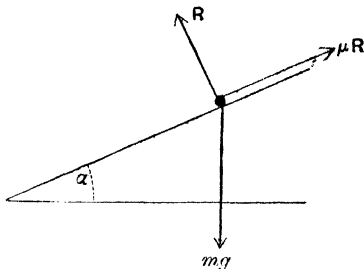
Similarly, if the particle were projected up the plane, we have to change the sign of μ , and its acceleration in a direction opposite to that of its motion is

$$g(\sin \alpha + \mu \cos \alpha).$$

190. Ex. *A train, of mass 50 tons, is ascending an incline of 1 in 100; the engine exerts a constant tractive force equal to the weight of 1 ton, and the resistance due to friction etc. may be taken at 8 lbs. weight per ton; find the acceleration with which the train ascends the incline.*

The train is retarded by the resolved part of its weight down the incline, and by the resistance of friction.

The latter is equal to 8×50 or 400 lbs. wt.



The incline is at an angle α to the horizon, where $\sin \alpha = \frac{1}{100}$.

The resolved part of the weight down the incline therefore

$$= W \sin \alpha = 50 \times 2240 \times \frac{1}{100} \text{ lbs. wt.} \\ = 1120 \text{ lbs. wt.}$$

Hence the total force to retard the train = 1520 lbs. wt.

But the engine pulls with a force equal to 2240 lbs. weight.

Therefore the total force to increase the speed equals (2240 - 1520) or 720 lbs. weight, *i.e.* 720*g* poundals.

Also the mass moved is 50×2240 lbs.

$$\text{Hence the acceleration} = \frac{720g}{50 \times 2240} \\ = \frac{9g}{1400} \text{ ft.-sec. units.}$$

Since the acceleration is known, we can, by Art. 156, find the velocity acquired, and the space described, in a given time, *etc.*

EXAMPLES. XXX

1. A body is projected with a velocity of 80 feet per second up a smooth inclined plane, whose inclination is 30° ; find the space described, and the time that elapses, before it comes to rest.

2. A heavy particle slides from rest down a smooth inclined plane which is 15 feet long and 12 feet high. What is its velocity when it reaches the ground, and how long does it take?

3. A particle sliding down a smooth plane, 16 feet long, acquires a velocity of $16\sqrt{2}$ feet per second; find the inclination of the plane to the horizon.

4. What is the ratio of the height to the length of a smooth inclined plane, so that a body may be four times as long in sliding down the plane as in falling freely down the height of the plane starting from rest?

5. A heavy body slides from rest down a smooth plane inclined at 30° to the horizon. How many seconds will it be in sliding 240 feet down the plane, and what will be its velocity when it has described this distance?

6. A particle slides without friction down an inclined plane, and in the 5th second after starting passes over a distance of 2207.25 centimetres; find the inclination of the plane to the horizon.

7. A particle, of mass 5 lbs., is placed on a smooth plane inclined at 30° to the horizon and connected by a string passing over the top of the plane with a particle of mass 3 lbs. which hangs vertically; find (1) the common acceleration, (2) the tension of the string, (3) the velocity at the end of 3 seconds, and (4) the space described in 3 seconds.

8. A body, of mass 12 lbs., is placed on an inclined plane, whose height is half its length, and is connected by a light string passing over a pulley at the top of the plane with a mass of 8 lbs. which hangs freely; find the distance described by the masses in 5 seconds.

9. A mass of 6 ounces slides down a smooth inclined plane, whose height is half its length, and draws another mass from rest over a distance of 3 feet in 5 seconds along a horizontal table which is level with the top of the plane over which the string passes; find the mass on the table.

✓ 10. If a train of 200 tons, moving at the rate of 30 miles per hour, can be stopped in 60 yards, compare the friction with the weight of a ton.

11. A train is running on horizontal rails at the rate of 30 miles per hour, the resistance due to friction, etc. being 10 lbs. wt. per ton; if the steam be shut off, find (1) the time that elapses before the train comes to rest, and (2) the distance described in this time.

12. In the previous question if the train is ascending an incline of 1 in 112, find the corresponding time and distance.

13. A train of mass 200 tons is running at the rate of 40 miles per hour down an incline of 1 in 120; find the resistance necessary to stop it in half a mile.

14. A train runs from rest for 1 mile down a plane whose descent is 1 foot vertically for each 100 feet of its length; if the resistances be equal to 8 lbs. per ton, how far will the train be carried along the horizontal level at the foot of the incline?

15. A train of mass 140 tons, travelling at the rate of 15 miles per hour, comes to the top of an incline of 1 in 128, the length of the incline being half a mile, and steam is then shut off; taking the resistance due to friction, etc. as 10 lbs. wt. per ton, find the distance it describes on a horizontal line at the foot of the incline before coming to rest.

If on arriving at the foot of the incline a brake-van, of weight 10 tons, have all its wheels prevented from revolving, find the distance described by the train, assuming the coefficient of friction between the wheels and the line to be .5.

16. A mass of 5 lbs. on a rough horizontal table is connected by a string with a mass of 8 lbs. which hangs over the edge of the table; if the coefficient of friction be $\frac{1}{2}$, find the resultant acceleration.

Find also the coefficient of friction if the acceleration be half that of a freely falling body.

17. A mass of 20 lbs. is moved along a rough horizontal table by means of a string which is attached to a mass of 4 lbs. hanging over the edge of the table; if the masses take twice the time to acquire the same velocity from rest that they do when the table is smooth, find the coefficient of friction.

18. A body, of mass 10 lbs., is placed on a rough plane, whose coefficient of friction is $\frac{1}{\sqrt{3}}$ and whose inclination to the horizon is 30° ; if the length of the plane be 4 feet and the body be acted on by a force, parallel to the plane, equal to 15 lbs. weight, find the time that elapses before it reaches the top of the plane and its velocity there.

19. If in the previous question the body be connected with a mass of 15 lbs., hanging freely, by means of a string passing over the top of the plane, find the time and velocity.

20. A particle slides down a rough inclined plane, whose inclination to the horizon is 45° and whose coefficient of friction is $\frac{3}{4}$; shew that the time of descending any space is twice what it would be if the plane were perfectly smooth.

191. *A body, of mass m lbs., is placed on a horizontal plane which is in motion with a vertical upward acceleration f ; find the reaction between the body and the plane.*

Let R be the reaction between the body and the plane.

Since the acceleration is vertically upwards, the total force acting on the body must be vertically upwards.

The only force, besides R , acting on the body is its weight mg acting vertically downwards.

Hence the total force is $R - mg$ vertically upwards, and this produces an acceleration f ; hence

$$R - mg = mf, \text{ giving } R.$$

In a similar manner it may be shewn that, if the body be moving with a downward acceleration f , the reaction R_1 is given by

$$mg - R_1 = mf.$$

We note that the reaction is greater or less than the weight of the body, according as the acceleration of the body is upwards or downwards.

Ex. 1. *A body, of mass 20 lbs., is moving with (1) an upward acceleration of 12 ft.-sec. units, (2) a downward acceleration of the same magnitude; find the reactions.*

In the first case we have

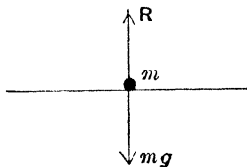
$$R - 20 \cdot g = 20 \cdot 12.$$

$$\therefore R = 20(32 + 12) \text{ poundals} = \text{wt. of } 27\frac{1}{2} \text{ lbs.}$$

In the second case we have

$$20 \cdot g - R_1 = 20 \cdot 12.$$

$$\therefore R = 20(32 - 12) \text{ poundals} = \text{wt. of } 12\frac{1}{2} \text{ lbs.}$$



Ex. 2. Two scale-pans, each of mass 3 ounces, are connected by a light string passing over a smooth pulley. If masses of 4 and 6 ounces respectively be placed in the pans, find the reactions of the pans during the subsequent motion.

On one side the total mass will be 9 ounces and on the other side 7 ounces. Hence, by Art. 184, the acceleration $f = \frac{9-7}{9+7}g = \frac{g}{8}$.

Let P poundals be the reaction on the 4 oz. mass. The total force on this 4 oz. mass therefore $= P - \frac{4}{16}g$ poundals upwards,

so that
$$P - \frac{4}{16}g = \frac{4}{16}f;$$

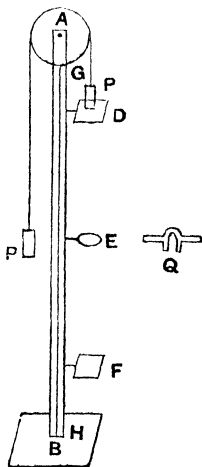
$$\therefore P = \frac{4}{16}(g+f) = \frac{4}{16} \cdot \frac{9}{8}g = \text{weight of } 4\frac{1}{2} \text{ ounces.}$$

If P' poundals be the reaction on the other mass, the total force on it is $\frac{6}{16}g - P'$ downwards,

$$\therefore \frac{6}{16}g - P' = \frac{6}{16}f.$$

$$\therefore P' = \frac{6}{16} \cdot \frac{7}{8}g = 5\frac{1}{4} \text{ ounces weight.}$$

192. Atwood's Machine. This machine is used to verify the laws of motion, and to obtain an approximate value for g . In its simplest form it consists of a vertical pillar AB , of about 8 feet in height, firmly clamped to the ground, and carrying at its top a light pulley which will move very freely. This pillar is graduated and carries two platforms, D and F , and a ring E , all of which can be affixed by screws at any height desired. The platform D can also be instantaneously dropped. Over the pulley passes a fine cord supporting at its ends two long thin equal weights, one of which, P , can freely pass through the ring E . Another small weight Q is provided, which can be laid upon the weight P , but which cannot pass through the ring E .



The weight Q is laid upon P and

the platform D is dropped and motion ensues; the weight Q is left behind as the weight P passes through the ring; the weight P then traverses the distance EF with constant velocity, and the time T which it takes to describe this distance is carefully measured.

By Art. 184 the acceleration of the system as the weight falls from D to E is

$$\frac{(Q+P)-P}{(Q+P)+P}g, \text{ i.e. } \frac{Q}{Q+2P}g.$$

Denote this by f , and let $DE = h$.

Then the velocity v on arriving at E is given by

$$v^2 = 2fh.$$

After passing E , the distance EF is described with constant velocity v .

Hence, if $EF = h_1$, we have

$$T = \frac{h_1}{v} = \frac{h_1}{\sqrt{2fh}}, \text{ so that } h_1^2 = \frac{2Q}{Q+2P}ghT^2.$$

Since all the quantities involved are known, this relation gives us the value of g .

By giving different values to P , Q , h , and h_1 , we can in this manner verify all the fundamental laws of motion.

In practice, the value of g cannot by this method be found to any great degree of accuracy and the interest of Atwood's machine is chiefly of an antiquarian character; the chief causes of discrepancy are the mass of the pulley, which cannot be neglected, the friction of the pivot on which the wheel turns, and the resistance of the air. It is also difficult accurately to measure the times involved in the experiment.

The friction of the pivot may be minimised if its ends do not rest on fixed supports, but on the circumferences of four light wheels, called friction wheels, two on each side, which turn very freely.

There are other pieces of apparatus for securing the accuracy of the experiment as far as possible, *e.g.* for instantaneously withdrawing the platform D at the required moment.

198. By using Atwood's machine, to shew that the acceleration of a given mass is proportional to the force acting on it.

We shall assume that the statement is true and see whether the results we deduce therefrom are verified by experiment.

To explain the method of procedure we shall take a numerical example.

Let P be $49\frac{1}{2}$ ozs. and Q 1 oz. so that the mass moved is 100 ozs. and the moving force is the weight of 1 oz.

The acceleration of the system therefore $= \frac{1}{100}g$ (Art. 184).

Let the distance DE be one foot, so that the velocity when Q is taken off $= \sqrt{2 \cdot \frac{g}{100} \cdot 1} = \frac{8}{10}$ ft. per sec., if, for simplicity, we take g equal to 32.

Let the platform F be carefully placed at such a point that the mass will move from E to F in some definite time, say 2 secs.

Then $EF = \frac{8}{10} \cdot 2 = \frac{8}{5}$ feet.

Now alter the conditions. Make P equal to 48 and Q equal to 4 ozs. The mass moved is still 100 ozs. and the moving force is now the weight of 4 ozs.

The acceleration is now $\frac{4g}{100}$ and the velocity at E

$$= \sqrt{2 \cdot \frac{4g}{100} \cdot 1} = \frac{8}{5} \text{ ft. per second.}$$

In 2 seconds the mass would now describe $\frac{16}{5}$ feet, so that, if our hypothesis be correct, the platform F must be twice as far from E as before. *This is found on trial to be correct.*

Similarly if we make $P = 45\frac{1}{2}$ ozs. and $Q = 9$ ozs., so that the mass moved is still 100 ozs., the theory would give us that EF should be $2\frac{4}{5}$ feet, and this would be found to be correct.

The experiment should now be tried over again *ab initio* and P and Q be given different values from the above; alterations should then be made in their values so that $2P + Q$ is constant.

By the same method, to shew that the force varies as the mass when the acceleration is constant.

As before let $P = 49\frac{1}{2}$ ozs. and $Q = 1$ oz. so that, as in the last experiment, we have $EF = \frac{8}{5}$ feet.

Secondly, let $P = 99$ ozs. and $Q = 2$ ozs., so that the moving force is doubled and the mass moved is doubled. Hence, if our enunciation be correct, the acceleration should be the same, since

$$\frac{\text{second moving force}}{\text{second mass moved}} = \frac{\text{first moving force}}{\text{first mass moved}}.$$

The distance EF moved through in 2 seconds should therefore be the same as before, and this, on trial, is found to be the case.

Similarly if we make $P=148\frac{1}{2}$ ozs. and $Q=3$ ozs. the same result would be found to follow.

In actual practice some extra weight R must be put on in order to overcome the friction at the pulley, etc. This should be determined in each case before Q is put on; it will be that weight which will just make the P on which it is placed move very slowly and uniformly downwards. This weight R must be kept on when Q is added and must not be counted as part of Q in the above work.

EXAMPLES. XXXI

1. If I jump off a table with a twenty-pound weight in my hand, what is the pressure of the weight on my hand?

2. A mass of 20 lbs. rests on a horizontal plane which is made to ascend (1) with a constant velocity of 1 foot per second, (2) with a constant acceleration of 1 foot per second per second; find in each case the reaction of the plane.

3. A man, whose mass is 8 stone, stands on a lift which moves with a uniform acceleration of 12 ft.-sec. units; find the reaction of the floor when the lift is (1) ascending, and (2) descending.

4. A bucket containing 1 cwt. of coal is drawn up the shaft of a coal-pit, and the pressure of the coal on the bottom of the bucket is equal to the weight of 126 lbs. Find the acceleration of the bucket.

5. A balloon ascends with a uniformly accelerated velocity, so that a mass of 1 cwt. produces on the floor of the balloon the same pressure which 116 lbs. would produce on the earth's surface; find the height which the balloon will have attained in one minute from the time of starting.

6. Two scale-pans, each of mass 2 ounces, are suspended by a weightless string passing over a smooth pulley; a mass of 10 ounces is placed in the one; and 4 ounces in the other. Find the tension of the string and the reactions of the scale-pans.

7. A string, passing over a smooth pulley, supports two scale-pans at its ends, the mass of each scale-pan being 1 ounce. If masses of 2 and 4 ounces respectively be placed in the scale-pans, find the acceleration of the system, the tension of the string, and the reactions between the masses and the scale-pans.

8. The two masses in an Atwood's machine are each 240 grammes, and an additional mass of 10 grammes being placed on one of them it is observed to descend through 10 metres in 10 seconds; hence shew that $g=980$.

9. Explain how to use Atwood's machine to shew that a body acted on by a constant force moves with constant acceleration.

CHAPTER XV

IMPULSE, WORK, AND ENERGY

194. Impulse. Def. *The impulse of a force in a given time is equal to the product of the force (if constant, and the mean value of the force if variable) and the time during which it acts.*

The impulse of a force P acting for a time t is therefore $P.t$.

The impulse of a force is also equal to the momentum generated by the force in the given time. For suppose a particle, of mass m , moving initially with velocity u is acted on by a constant force P for time t . If f be the resulting acceleration, we have $P = mf$.

But, if v be the velocity of the particle at the end of time t , we have

$$v = u + ft.$$

Hence the impulse $= Pt = mft = mv - mu$

$=$ the momentum generated in the given time.

The same result is also true if the force be variable.

Hence it follows that the second law of motion might have been enunciated in the following form:

The change of momentum of a particle in a given time is equal to the impulse of the force which produces it and is in the same direction.

195. Impulsive Forces. Suppose we have a force P acting for a time τ on a body whose mass is m , and let the velocities of the mass at the beginning and end of this time be u and v . Then by the last article

$$P\tau = m(v - u).$$

Let now the force become bigger and bigger, and the time τ smaller and smaller. Then ultimately P will be almost infinitely big and τ almost infinitely small, and yet their product *may* be finite. For example P may be equal to 10^7 poundals, τ equal to $\frac{1}{10^7}$ seconds, and m equal to one

pound, in which case the change of velocity produced is the unit of velocity.

To find the whole effect of a finite force acting for a finite time we have to find two things, (1) the change in the velocity of the particle produced by the force during the time it acts, and (2) the change in the position of the particle during this time. Now in the case of an infinitely large force acting for an infinitely short time, the body moves only a very short distance whilst the force is acting, so that this change of position of the particle may be neglected. Hence the total effect of such a force is known when we know the change of momentum which it produces.

Such a force is called an impulsive force. Hence

Def. *An impulsive force is a very great force acting for a very short time, so that the change in the position of the particle during the time the force acts on it may be neglected. Its whole effect is measured by its impulse, or the change of momentum produced.*

In actual practice we never have any experience of an infinitely great force acting for an infinitely short time. Approximate examples are, however, the blow of a hammer, and the collision of two billiard balls.

The above will be true even if the force be not uniform. In the ordinary case of the collision of two billiard balls the force generally varies very considerably.

Ex. 1. A body, whose mass is 9 lbs., is acted on by a force which changes its velocity from 20 miles per hour to 30 miles per hour. Find the impulse of the force.

Ans. 132 units of impulse.

Ex. 2. A mass of 2 lbs. at rest is struck and starts off with a velocity of 10 feet per second; assuming the time during which the blow lasts to be one-hundredth of a second, find the average value of the force acting on the mass.

Ans. 2000 poundals.

Ex. 3. A glass marble, whose mass is 1 ounce, falls from a height of 25 feet, and rebounds to a height of 16 feet; find the impulse, and the average force between the marble and the floor if the time during which they are in contact be one-tenth of a second.

Ans. $4\frac{1}{2}$ units of impulse; 47 poundals.

196. Impact of two bodies. When two masses A and B impinge, then, by the third law of motion, the action of A on B is, at each instant during which they are in contact, equal and opposite to that of B on A .

Hence the impulse of the action of A on B is equal and opposite to the impulse of the action of B on A .

It follows that the change in the momentum of B is equal and opposite to the change in the momentum of A , and therefore the sum of these changes, measured in the same direction, is zero.

Hence the sum of the momenta of the two masses, measured in the same direction, is unaltered by their impact.

Ex. 1. *A body, of mass 3 lbs., moving with velocity 13 feet per second overtakes a body, of mass 2 lbs., moving with velocity 3 feet per second in the same straight line, and they coalesce and form one body; find the velocity of this single body.*

Let V be the required velocity. Then, since the sum of the momenta of the two bodies is unaltered by the impact, we have

$$(3+2)V = 3 \times 13 + 2 \times 3 = 45 \text{ units of momentum.}$$

$$\therefore V = 9 \text{ ft. per sec.}$$

Ex. 2. *If in the last example the second body be moving in the direction opposite to that of the first, find the resulting velocity.*

In this case the momentum of the first body is represented by 3×13 and that of the second by -2×3 . Hence, if V_1 be the required velocity, we have

$$(3+2)V_1 = 3 \times 13 - 2 \times 3 = 33 \text{ units of momentum.}$$

$$\therefore V_1 = \frac{33}{5} = 6\frac{3}{5} \text{ ft. per sec.}$$

197. Motion of a shot and gun. When a gun is fired, the powder is almost instantaneously converted into a gas at a very high pressure, which by its expansion forces the shot out. The action of the gas is similar to that of a compressed spring trying to recover its natural position. The force exerted on the shot forwards is, at any instant before the shot leaves the gun, equal and opposite to that exerted on the gun backwards, and therefore the impulse of this force on the shot is equal and opposite to the impulse of the force on the gun. Hence the momentum generated in the shot is equal and opposite to that generated in the gun, if the latter be free to move.

Ex. A shot, whose mass is 400 lbs., is projected from a gun, of mass 50 tons, with a velocity of 900 feet per second; find the resulting velocity of the gun.

Since the momentum of the gun is equal and opposite to that of the shot we have, if v be the velocity communicated to the gun,

$$50 \times 2240 \times v = 400 \times 900.$$

$$\therefore v = 3\frac{3}{14} \text{ ft. per sec.}$$

EXAMPLES XXXII

1. A body, of mass 7 lbs., moving with a velocity of 10 feet per second, overtakes a body, of mass 20 lbs., moving with a velocity of 2 feet per second in the same direction as the first; if after the impact they move forward with a common velocity, find its magnitude.

2. A body, of mass 8 lbs., moving with a velocity of 6 feet per second, overtakes a body, of mass 24 lbs., moving with a velocity of 2 feet per second in the same direction as the first; if after the impact they coalesce into one body, shew that the velocity of the compound body is 3 feet per second.

If they were moving in opposite directions, shew that after impact the compound body is at rest.

3. A body, of mass 10 lbs., moving with a velocity of 4 feet per second meets a body, of mass 12 lbs., moving in the opposite direction with a velocity of 7 feet per second; if they coalesce into one body, shew that it will have a velocity of 2 feet per second in the direction in which the larger body was originally moving.

4. A shot, of mass 1 ounce, is projected with a velocity of 1000 feet per second from a gun of mass 10 lbs.; find the velocity with which the latter begins to recoil.

5. A shot of 800 lbs. is projected from a 40-ton gun with a velocity of 2000 feet per second; find the velocity with which the gun would commence to recoil, if free to move in the line of projection.

6. A shot, of mass 700 lbs., is fired with a velocity of 1700 feet per second from a gun of mass 38 tons; if the recoil be resisted by a constant pressure equal to the weight of 17 tons, through how many feet will the gun recoil?

7. A gun, of mass 1 ton, fires a shot of mass 28 lbs. and recoils up a smooth inclined plane, rising to a height of 5 feet; find the initial velocity of the shot.

8. A hammer, of mass 4 cwt., falls through 4 feet and comes to rest after striking a mass of iron, the duration of the blow being $\frac{1}{60}$ th of a second; find the pressure, supposing it to be uniform, which is exerted by the hammer on the iron.

9. Masses m and $2m$ are connected by a string passing over a smooth pulley; at the end of 3 seconds a mass m is picked up by the ascending body; find the resulting motion.

10. One inch of rain fell in 5 hours. Assuming that the drops fall freely from a height of 300 yds., find the pressure on the ground per square mile due to the rain during the storm, the mass of a cubic foot of water being 1000 ounces, and the rain being uniform and continuous.

The amount of rain that falls on a square foot during the storm is $\frac{1}{12}$ of a cubic foot, and its mass is $\frac{250}{3}$ ounces.

Hence the mass that falls per second

$$= \frac{250}{3 \times 16} \times \frac{1}{5 \cdot 60 \cdot 60} = \frac{1}{48 \times 72} \text{ lbs. per sq. foot.}$$

The velocity of each raindrop on touching the ground is

$$\sqrt{2 \times g \times 300 \times 3}, \text{ or } 240 \text{ ft. per second.}$$

Therefore the momentum that is destroyed per second is

$$\frac{1}{48 \times 72} \times 240, \text{ or } \frac{5}{72} \text{ units of momentum.}$$

But the number of units of momentum destroyed per second is equal to the number of poundals in the acting force (Art. 173). Hence the pressure on the ground per sq. foot = $\frac{5}{72}$ poundals.

Hence the pressure per square mile

$$= \text{weight of } 9 \times 4840 \times 640 \times \frac{5}{32 \times 72} \text{ lbs.}$$

$$= \text{weight of 27 tons approximately.}$$

11. Find the pressure in lbs. wt. per acre due to the impact of a fall of rain of 3 inches in 24 hours, supposing the rain to have a velocity due to falling freely through 400 feet.

12. A jet of water is projected against a wall so that 300 gallons strike the wall per second with a horizontal velocity of 80 feet per second. Assuming that the mass of a gallon of water is 10 lbs., find the reaction of the wall in pounds weight.

198. Work. We have pointed out in Art. 135, that the unit of work used by engineers is a Foot-Pound, which is the work done in raising the weight of one pound through one foot.

The British absolute unit of work is the work done by a poundal in moving its point of application through one foot.

This unit of work is called a **Foot-Poundal**.

With this unit of work the work done by a force of P poundals in moving its point of application through s feet is $P \cdot s$ foot-poundals.

Since the weight of a pound is equal to g -poundals, it follows that a Foot-Pound is equal to g Foot-Poundals.

The c.g.s. unit of work is that done by a dyne in moving its point of application through a centimetre, and is called an **Erg**.

$$\frac{\text{A Foot-Poundal}}{\text{An Erg}} = \frac{\text{Poundal} \times \text{Foot}}{\text{Dyne} \times \text{Centimetre}} = \frac{981}{32 \cdot 2} \times 453 \cdot 6 \times \frac{12}{\cdot 3937} \text{ nearly}$$

(Arts. 178 and 151), so that one Foot-Poundal = 421,390 Ergs nearly.

199. Ex. 1. *What is the H.P. of an engine which keeps a train, of mass 150 tons, moving at a uniform rate of 60 miles per hour, the resistances to the motion due to friction, the resistance of the air, etc. being taken at 10 lbs. weight per ton?*

The force to stop the train is equal to the weight of 150×10 , i.e. 1500, lbs. weight.

Now 60 miles per hour is equal to 88 feet per second.

Hence a force, equal to 1500 lbs. wt., has its point of application moved through 88 feet in a second, and hence the work done is 1500×88 foot-pounds per second.

If x be the H.P. of the engine, the work it does per minute is $x \times 33000$ foot-lbs., and hence the work per second is $x \times 550$ foot-lbs.

$$\therefore x \times 550 = 1500 \times 88, \text{ so that } x = 240.$$

Ex. 2. *Find the least H.P. of an engine which is able in 4 minutes to generate in a train, of mass 100 tons, a velocity of 30 miles per hour on a level line, the resistances due to friction, etc. being equal to 8 lbs. weight per ton, and the pull of the engine being assumed constant.*

Since in 240 seconds a velocity of 44 feet per second is generated the acceleration of the train must be $\frac{44}{240}$ or $\frac{11}{60}$ foot-second unit.

Let the force exerted by the engine be P poundals.

The resistance due to friction is equal to 800 pounds weight; hence the total force on the train is $P - 800g$ poundals.

$$\text{Hence} \quad P - 800g = 100 \times 2240 \times \frac{11}{60}.$$

$$\begin{aligned} \therefore P &= 800 \left(g + \frac{154}{3} \right) \text{ poundals} = 800 \left(1 + \frac{154}{3 \times 32} \right) \text{ lbs. weight} \\ &= 800 \times \frac{125}{48} \text{ lbs. weight.} \end{aligned}$$

When the train is moving at the rate of 30 miles per hour, i.e. 44 feet per second, the work done per second must be $800 \times \frac{125}{48} \times 44$ foot-lbs.

Hence, if x be the H.P. of the engine, we have

$$x \times 550 = 800 \times \frac{125}{48} \times 44, \text{ so that } x = 166\frac{2}{3}.$$

EXAMPLES. XXXIII

1. A train, of mass 50 tons, is kept moving at the uniform rate of 30 miles per hour on the level, the resistances of air, friction, etc., being 40 lbs. weight per ton. Find the H.P. of the engine.

2. What is the horse-power of an engine which keeps a train going at the rate of 40 miles per hour against a resistance equal to 2000 lbs. weight?

3. A train, of mass 100 tons, travels at 40 miles per hour up an incline of 1 in 200. Find the H.P. of the engine that will draw the train, neglecting all resistances except that of gravity.

4. A train of mass 200 tons, including the engine, is drawn up an incline of 3 in 500 at the rate of 40 miles per hour by an engine of 600 H.P.; find the resistances per ton due to friction, etc.

5. Find the H.P. of an engine which can travel at the rate of 25 miles per hour up an incline of 1 in 100, the mass of the engine and load being 10 tons, and the resistances due to friction, etc. being 10 lbs. weight per ton.

6. Determine the rate in H.P. at which an engine must be able to work in order to generate a velocity of 20 miles per hour on the level in a train of mass 60 tons in 3 minutes after starting, the resistances to the motion being taken at 10 lbs. per ton and the force exerted by the train being assumed to be constant.

7. Find the work done by gravity on a stone having a mass of $\frac{1}{2}$ lb. during the tenth second of its fall from rest.

8. A steamer with engines of 25,000 H.P. can be just kept going at the rate of 20 miles per hour. Shew that the resistance of the water is equal to 209·2... tons wt.

***200. Energy. Def.** *The Energy of a body is its capacity for doing work and is of two kinds, Kinetic and Potential.*

The Kinetic Energy of a body is the energy which it possesses by virtue of its motion, and is measured by the amount of work that the body can perform against the impressed forces before its velocity is destroyed.

A falling body, a swinging pendulum, a revolving fly-wheel, and a cannon-ball in motion all possess kinetic energy.

Consider the case of a particle, of mass m , moving with velocity u , and let us find the work done by it before it comes to rest.

Suppose it brought to rest by a constant force P resisting its motion, which produces in it an acceleration $-f$ given by $P = mf$.

Let x be the space described by the particle before it comes to rest, so that $0 = u^2 + 2(-f) \cdot x$.

$$\therefore fx = \frac{1}{2}u^2.$$

Hence the kinetic energy of the particle
 = work done by it before it comes to rest
 = $Px = mfx = \frac{1}{2}mu^2$.

Hence the kinetic energy of a particle is equal to the product of its mass and one half the square of its velocity.

***201. Theorem.** *To shew that the change of kinetic energy per unit of space is equal to the acting force.*

If a force P , acting on a particle of mass m , change its velocity from u to v in time t whilst the particle moves through a space s , we have $v^2 - u^2 = 2fs$, where f is the acceleration produced.

$$\therefore \frac{\frac{1}{2}mv^2 - \frac{1}{2}mu^2}{s} = mf = P \dots \dots \dots (1).$$

This equation proves the proposition when the force is constant.

Cor. It follows from equation (1) that the change in the kinetic energy of a particle is equal to the work done on it.

***202. The Potential Energy of a body is the work it can do by means of its position in passing from its present configuration to some standard configuration (usually called its zero position).**

A bent spring (*e.g.* a watch-spring, which, by its uncoiling, keeps a watch going) has potential energy, viz. the work it can do in recovering its natural shape. A body raised to a height above the ground (*e.g.* a clock-weight when the clock is wound up, a stone at the edge of a precipice, or water stored up in a reservoir) has potential energy, viz. the work its weight can do as it falls to the earth's surface. Compressed air has potential energy, viz. the work it can do in expanding to the volume it would occupy in the atmosphere.

***203.** *A particle of mass m falls from rest at a height h above the ground; to shew that the sum of its potential and kinetic energies is constant throughout the motion.*

Let H be the point from which the particle starts, and O the point where it reaches the ground.

Let v be its velocity when it has fallen through a distance $HP (= x)$, so that $v^2 = 2gx$.

Its kinetic energy at $P = \frac{1}{2}mv^2 = mgx$.

Also its potential energy at P

$=$ the work its weight can do as it falls from P to O

$= mg \cdot OP = mg(h - x)$.

Hence the sum of its kinetic and potential energies at P

$$= mgh.$$

But its potential energy when at H is mgh , and its kinetic energy there is zero.

Hence the sum of the potential and kinetic energies is the same at P as at H ; and, since P is *any* point, it follows that the sum of these two quantities is the same throughout the motion.

As the particle falls to the ground the potential energy, which it has when at its highest point, becomes transformed into kinetic energy, until the particle reaches the ground, when its store of potential energy becomes exhausted.

***204.** The statement proved in the previous article, viz., that the sum of the kinetic and potential energies is constant throughout the motion, is found to be true for all cases of motion where there is no friction or resistance of the air, nor any impacts. This is an elementary illustration of the Principle known as that of the **Conservation of Energy**, which states that Energy is indestructible. It may be changed into different forms but can never be destroyed. When a body slides along a rough plane some of its mechanical energy becomes transformed and reappears in the form of heat partly in the moving body and partly in the plane.

Ex. 1. A bullet, of mass 4 ozs., is fired into a target with a velocity of 1200 feet per second. The mass of the target is 20 lbs. and it is free to move; find the loss of kinetic energy in foot-pounds.

Let V be the resulting common velocity of the shot and target. Since no momentum is lost (Art. 196) we have

$$\left(20 + \frac{4}{16}\right)V = \frac{4}{16} \times 1200, \text{ so that } V = \frac{400}{27}.$$

The original kinetic energy $= \frac{1}{2} \cdot \frac{4}{16} \cdot 1200^2 = 180000$ foot-pounds.

The final kinetic energy $= \frac{1}{2} \left(20 + \frac{4}{16} \right) V^2 = \frac{20000}{9}$ foot-pounds.

The energy lost $= 180000 - \frac{20000}{9} = \frac{1600000}{9}$ foot-pounds
 $= \frac{50000}{9}$ ft.-lbs.

It will be noted that, in this case, although no momentum is lost by the impact, yet $\frac{80}{81}$ ths of the energy is destroyed.

It will be found that, in all cases of impact, kinetic energy is lost or rather is transformed.

Ex. 2. Compare the kinetic energies of the shot and gun in the example of Art. 197.

The kinetic energy of the shot $= \frac{1}{2} \cdot 400 \times (900)^2$ foot-pounds

$$= \frac{200 \times 900^2}{32} \text{ ft.-lbs.} = \frac{200 \times 900^2}{32 \times 2240} \text{ ft.-tons} \\ = 2260 \text{ ft.-tons nearly.}$$

The kinetic energy of the gun

$$= \frac{1}{2} \cdot 50 \times 2240 \times \left(\frac{45}{14} \right)^2 \text{ foot-pounds} \\ = \frac{25}{32} \times \left(\frac{45}{14} \right)^2 \text{ ft.-tons} = 8.07 \text{ ft.-tons nearly.}$$

The kinetic energy of the shot is thus 280 times that of the gun, although their momenta are equal.

It is to this great superiority in kinetic energy of the shot that its destructive power is due.

When we take into account the energy which has been transformed into heat, sound, light and other forms which modern Physics recognises as forms of energy, we find that there is no real loss of energy in an isolated system which is left to itself. This doctrine of the indestructibility of energy is the central Principle of Modern Science. It may be expressed thus:

Energy cannot be created nor can it be destroyed, but it may be transformed into any of the forms which it can take.

As a numerical illustration, it may be stated that 778 ft.-lbs. of work is equivalent to the heat necessary to raise the temperature of 1 lb. of water by 1° Fahrenheit, i.e. 778 ft.-lbs. is the **mechanical equivalent of heat**.

EXAMPLES. XXXIV

1. A body, of mass 10 lbs., is thrown vertically upwards with a velocity of 32 feet per second; what is its kinetic energy (1) at the moment of propulsion, (2) after half a second, (3) after one second?

2. Find the kinetic energy measured in foot-pounds of a cannon-ball of mass 25 pounds discharged with a velocity of 200 feet per second.

3. Find the kinetic energy in ergs of a cannon-ball of 10,000 grammes discharged with a velocity of 5000 centimetres per second.

4. What is the horse-power of an engine that can project 10,000 lbs. of water per minute with a velocity of 80 feet per second, twenty per cent. of the whole work being wasted by friction?

5. A cannon-ball, of mass 5000 grammes, is discharged with a velocity of 500 metres per second. Find its kinetic energy in ergs, and, if the cannon be free to move, and have a mass of 100 kilogrammes, find the energy of the recoil.

6. A bullet, of mass 2 ounces, is fired into a target with a velocity of 1280 feet per second. The mass of the target is 10 lbs. and it is free to move; find the loss of kinetic energy by the impact in foot-pounds.

7. Equal forces act for the same time upon unequal masses M and m ; what is the relation between (1) the momenta generated by the forces, and (2) the amounts of work done by them?

8. Find the H.P. of an engine that will in one hour raise 15,000 gallons of water from the bottom of a shaft 1100 feet deep, and deliver it with a velocity of 5 feet per second, assuming the mass of a gallon of water to be 10 lbs.

9. Find the H.P. exerted by an engine which works a pump which raises 2000 lbs. of water per minute through a height of 30 ft., the water flowing away with a velocity of 10 ft. per sec. Of the work done by the engine, shew that about 4.95 per cent. is employed in imparting to the water its velocity at efflux.

10. Compare (1) the momenta, and (2) the kinetic energies, of a bullet of mass 4 ozs. moving with a velocity of 1200 feet per second, and a cannon-ball of mass 15 lbs. moving with a velocity of 40 feet per second. Find the uniform forces which would bring each to rest in one second, and the distance through which each would move.

*CHAPTER XVI

COMPOSITION OF VELOCITIES AND ACCELERATIONS. PROJECTILES

205. SINCE the velocity of a point is known when its direction and magnitude are both known, we can conveniently represent the velocity of a moving point by a straight line AB ; thus, when we say that the velocities of two moving points are represented in magnitude and direction by the straight lines AB and CD , we mean that they move in directions parallel to the lines drawn from A to B , and C to D respectively, and with velocities which are proportional to the lengths AB and CD .

206. A body may have simultaneously velocities in two, or more, different directions. One of the simplest examples of this is when a person walks on the deck of a moving ship from one point of the deck to another. He has a motion with the ship, and one along the deck of the ship, and his motion in space is clearly different from what it would have been had either the ship remained at rest, or the man stayed at his original position on the deck.

Again, consider the case of a ship steaming with its bow pointing in a constant direction, say due north, whilst a current carries it in a different direction, say south-east, and suppose a sailor is climbing a vertical mast of the ship. The actual change of position and the velocity of the sailor clearly depend on three quantities, viz., the rate and direction of the ship's sailing, the rate and direction of the current, and the rate at which he climbs the mast. His actual velocity is said to be "compounded" of these three velocities.

In the following article we shew how to find the velocity which is equivalent to two velocities given in magnitude and direction.

207. Theorem. Parallelogram of Velocities.
If a moving point possess simultaneously velocities which are represented in magnitude and direction by the two sides of a

parallelogram drawn from a point, they are equivalent to a velocity which is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.

Let the two simultaneous velocities be represented by the lines AB and AC , and let their magnitudes be u and v .

Complete the parallelogram $BACD$.



Then we may imagine the motion of the point to be along the line AB with the velocity u , whilst the line AB moves parallel to the foot of the page so that its end A describes the line AC with velocity v . In the unit of time the moving point will have moved through a distance AB along the line AB , and the line AB will have in the same time moved into the position CD , so that at the end of the unit of time the moving point will be at D .

Now, since the two coexistent velocities are constant in magnitude and direction, the velocity of the point from A to D must also be constant in magnitude and direction; hence AD is the path described by the moving point in the unit of time.

Hence AD represents in magnitude and direction the velocity which is equivalent to the velocities represented by AB and AC .

To facilitate his understanding of the previous article, the student may look on AC as the direction of motion of a steamer, whilst AB is a chalked line, drawn along the deck of the steamer, along which a man is walking at a uniform rate.

208. Def. *The velocity which is equivalent to two or more velocities is called their **resultant** and these velocities are called the **components** of this resultant.*

Since the resultant of two velocities is found in the same way as the resultant of two forces, it can be shewn similarly, as in Art. 25, that the resultant of two velocities u and v acting at an angle α is

$$\sqrt{u^2 + v^2 + 2uv \cos \alpha}.$$

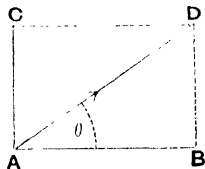
209. A velocity can be resolved into two component velocities in an infinite number of ways. For an infinite

number of parallelograms can be described having a given line AD as diagonal; and, if $ABDC$ be *any* one of these, the velocity AD is equivalent to the two component velocities AB and AC .

The most important case is when a velocity is to be resolved into two velocities in two directions *at right angles*, one of these directions being given. When we speak of the *component of a velocity in a given direction* it is understood that the other direction in which the given velocity is to be resolved is perpendicular to this given direction.

Thus, suppose we wish to resolve a velocity u , represented by AD , into two components at right angles to one another, one of these components being along a line AB making an angle θ with AD .

Draw DB perpendicular to AB , and complete the rectangle $ABDC$.



Then the velocity AD is equivalent to the two component velocities AB and AC .

Also $AB = AD \cos \theta = u \cos \theta$,
and $AC = BD = AD \sin \theta = u \sin \theta$.

We thus have the following important

Theorem. *A velocity u is equivalent to a velocity $u \cos \theta$ along a line making an angle θ with its own direction together with a velocity $u \sin \theta$ perpendicular to the direction of the first component.*

The case in which the angle θ is greater than a right angle may be considered as in Art. 27.

Ex. 1. A man is walking in a north-easterly direction with a velocity of 4 miles per hour; find the components of his velocity in directions due north and due east respectively.

Ans. Each is $2\sqrt{2}$ miles per hour.

Ex. 2. A point is moving in a straight line with a velocity of 10 feet per second; find the component of its velocity in a direction inclined at an angle of 30° to its direction of motion.

Ans. $5\sqrt{3}$ feet per second.

Ex. 3. A body is sliding down an inclined plane whose inclination to the horizontal is 60° ; find the components of its velocity in the horizontal and vertical directions.

Ans. $\frac{u}{2}$ and $u \frac{\sqrt{3}}{2}$, where u is the velocity of the body.

210. Triangle of Velocities. *If a moving point possess simultaneously velocities represented by the two sides AB and BC of a triangle taken in order, they are equivalent to a velocity represented by AC .*

For, completing the parallelogram $ABCD$, the lines AB and BC represent the same velocities as AB and AD and hence have as their resultant the velocity represented by AC .

Cor. If there be simultaneously impressed on a point three velocities represented by the sides of a triangle taken in order, the point will be at rest.

211. The resultant of two or more velocities is generally most conveniently found by resolving along two directions at right angles. The method is the same as that for forces in Art. 37.

Ex. 1. *A ship steams with its bow pointed due north with a velocity of 15 miles an hour, and is carried by a current which flows in a south-easterly direction at the rate of $3\sqrt{2}$ miles per hour. At the end of an hour find its distance and bearing from the point from which it started.*

The ship has two velocities, one being 15 miles per hour northwards, and the other $3\sqrt{2}$ miles per hour south-east.

Now the latter velocity is equivalent to

$3\sqrt{2} \cos 45^\circ$, that is, 3 miles per hour eastwards,
and $3\sqrt{2} \sin 45^\circ$, that is, 3 miles per hour southwards.

Hence the total velocity of the ship is 12 miles per hour northwards and 3 miles per hour eastwards.

Hence its resultant velocity is $\sqrt{12^2 + 3^2}$ or $\sqrt{153}$ miles per hour in a direction inclined at an angle, whose tangent is $\frac{1}{4}$, to the north, i.e., 12.37 miles per hour at $14^\circ 2'$ east of north.

Ex. 2. *A point possesses simultaneously velocities whose measures are 4, 3, 2, and 1; the angle between the first and second is 30° , between the second and third 90° , and between the third and fourth 120° ; find their resultant.*

Take OX along the direction of the first velocity and OY perpendicular to it.

The angles which the velocities make with OX are respectively 0° , 30° , 120° , and 240° .

Hence, if V be the resultant velocity inclined at an angle θ to OX , we have

$$V \cos \theta = 4 + 3 \cos 30^\circ + 2 \cos 120^\circ + 1 \cdot \cos 240^\circ,$$

$$\text{and } V \sin \theta = 3 \sin 30^\circ + 2 \sin 120^\circ + 1 \cdot \sin 240^\circ.$$

We therefore have

$$V \cos \theta = 4 + 3 \cdot \frac{\sqrt{3}}{2} + 2 \left(-\frac{1}{2} \right) + 1 \left(-\frac{1}{2} \right) = \frac{5 + 3\sqrt{3}}{2},$$

and $V \sin \theta = 3 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{3}}{2} - 1 \cdot \frac{\sqrt{3}}{2} = \frac{3 + \sqrt{3}}{2}.$

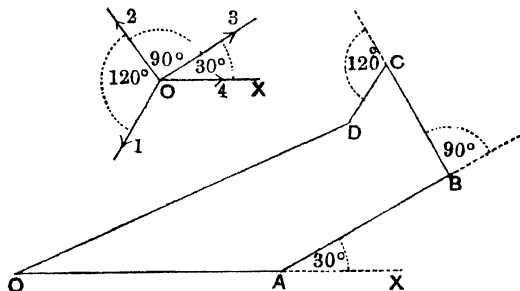
Hence, by squaring and adding,

$$V^2 = 16 + 9\sqrt{3} = 31.5885, \text{ so that } V = 5.62,$$

and, by division, $\tan \theta = \frac{3 + \sqrt{3}}{5 + 3\sqrt{3}} = 2\sqrt{3} - 3 = .4641 = \tan 24^\circ 54'.$

Hence the resultant is a velocity equal to 5.62 inclined at an angle $24^\circ 54'$ to the direction of the first velocity.

Graphically: This result may also be obtained by drawing; mark off OA on OX equal to 4 inches and draw AB , making AB equal to 3 inches and XAB equal to 30° .



Draw BC perpendicular to AB and equal to 2 inches, and then CD at an angle of 120° with BC produced and equal to 1 inch.

Join OD .

On measurement, $OD = 5.62$ inches, and the $\angle AOD = 25^\circ$ nearly.

EXAMPLES. XXXV

1. The velocity of a ship is $8\frac{2}{11}$ miles per hour, and a ball is bowled across the ship perpendicular to the direction of the ship with a velocity of 3 yards per second; describe the path of the ball in space and shew that it passes over 45 feet in 3 seconds.

2. A boat is rowed with a velocity of 6 miles per hour straight across a river which flows at the rate of 2 miles per hour. If its breadth be 300 feet, find how far down the river the boat will reach the opposite bank below the point at which it was originally directed.

3. A man wishes to cross a river to an exactly opposite point on the other bank; if he can pull his boat with twice the velocity of the current, find at what inclination to the current he must keep the boat pointed.

4. A boat is rowed on a river so that its speed in still water would be 6 miles per hour. If the river flow at the rate of 4 miles per hour, draw a figure to shew the direction in which the head of the boat must point so that the motion of the boat may be at right angles to the current.

5. A stream runs with a velocity of $1\frac{1}{2}$ miles per hour; find in what direction a swimmer, whose velocity is $2\frac{1}{2}$ miles per hour, should start in order to cross the stream perpendicularly.

What direction should be taken in order to cross in the shortest time?

6. A ship is steaming in a direction due north across a current running due west. At the end of one hour it is found that the ship has made $8\sqrt{3}$ miles in a direction 30° west of north. Find the velocity of the current, and the rate at which the ship is steaming.

7. A ship is sailing north at the rate of 4 feet per second; the current is taking it east at the rate of 3 feet per second, and a sailor is climbing a vertical pole at the rate of 2 feet per second; find the velocity and direction of the sailor in space.

8. A point which possesses velocities represented by 7, 8, and 13 is at rest; find the angle between the directions of the two smaller velocities.

9. A point possesses velocities represented by 3, 19, and 9 inclined at angles of 120° to one another; find their resultant.

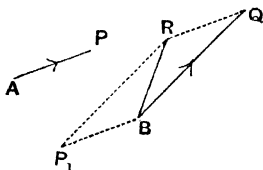
10. A point possesses simultaneously velocities represented by u , $2u$, $3\sqrt{3}u$, and $4u$; the angles between the first and second, the second and third, and the third and fourth, are respectively 60° , 90° , and 150° ; shew that the resultant is u in a direction inclined at an angle of 120° to that of the first velocity.

11. A point possesses simultaneously four velocities; the first is 24 ft. per sec.; the second is 36 ft. per sec. at 40° to the first; the third is 45 ft. per sec. at 50° to the second; and the fourth is 60 ft. per sec. at 35° to the third; shew, by a drawing, that the resultant velocity is about 118.5 ft. per sec. at about 82° to the direction of the first component velocity.

***212. Relative Velocity.** When two points, A and B , are in motion the relative velocity of B with respect to A is the velocity which B would *seem* to have to a man travelling with A , if the man were unconscious of his own motion. Now the relative motion of two points is unaltered by giving to each a common velocity; thus the relative motion of two persons on the deck of a steamer is the same whether the steamer is at rest, or moving quickly or slowly. Give therefore to both points, A and B , an additional velocity, which is equal and opposite to that of A , so that A is reduced to

rest. The resulting velocity of B is the required relative velocity of B with respect to A , and is thus obtained by *compounding with the velocity of B a velocity equal and opposite to that of A* .

Thus, if AP represent the velocity of A and BQ that of B , we draw BP_1 equal and opposite to AP , and complete the parallelogram BP_1RQ ; then BR represents the relative velocity of B with respect to A .



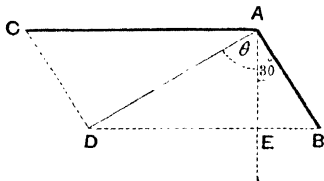
Ex. 1. A train is travelling along a horizontal rail at the rate of 30 miles per hour, and rain is driven by the wind, which is in the same direction as the motion of the train, so that it falls with a velocity of 22 feet per second and at an angle of 30° with the vertical. Find the apparent direction of the rain to a person travelling with the train.

The velocity of the train is 44 feet per second.

Let AB represent the actual velocity of the rain so that, if AE be a vertical line, the angle EAB is 30° .

Draw AC horizontal and opposite to the direction of the train and let it represent in magnitude the velocity, 44 feet per second, of the train.

Complete the parallelogram $ABDC$. Then AD is the apparent direction of the rain.



If $\angle EAD = \theta$, we have (Appendix I, Art. 14)

$$\frac{BD}{AB} = \frac{\sin DAB}{\sin BDA} = \frac{\sin (\theta + 30^\circ)}{\cos \theta}.$$

$$\therefore \frac{44}{22} = \frac{\sin \theta \cos 30^\circ + \cos \theta \sin 30^\circ}{\cos \theta} = \tan \theta \cos 30^\circ + \sin 30^\circ.$$

$$\therefore 2 = \tan \theta \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}.$$

$$\therefore \tan \theta = \sqrt{3} = \tan 60^\circ.$$

Hence θ is 60° . It follows, since BAD is a right angle, that the apparent direction of the rain is at right angles to its real direction.

Ex. 2. A ship, A , is sailing east at the rate of 7 miles per hour, and the relative velocity of a second ship, B , with respect to it is $5\sqrt{2}$ miles per hour south-east. Find the real velocity and direction of B .

Draw $OC (= 5\sqrt{2})$ south-east and OD west and equal to 7. Complete the parallelogram $ODCE$. Then OE represents the real velocity of B . For OE , with OD a velocity equal and opposite to that of A , gives OC

as resultant. On measurement, OE is equal to 13 and is in the east-south-east direction nearly.

Or, by calculation, let OE be V at an angle θ to the east; then, on resolving,

$$V \cos \theta - 7 = 5\sqrt{2} \cos 45^\circ = 5,$$

and

$$V \sin \theta = 5\sqrt{2} \sin 45^\circ = 5.$$

Hence $V = 13$, and $\tan \theta = \frac{5}{12} = .41667$, so that $\theta = 22^\circ 37'$.

EXAMPLES. XXXVI

1. A railway train, moving at the rate of 30 miles per hour, is struck by a stone, moving horizontally and at right angles to the train with a velocity of 33 feet per second. Find the magnitude and direction of the velocity with which the stone appears to meet the train.

2. One ship is sailing due east at the rate of 12 miles per hour, and another ship is sailing due north at the rate of 16 miles per hour; find the relative velocity of the second ship with respect to the first.

3. One ship is sailing south with a velocity of $15\sqrt{2}$ miles per hour, and another south-east at the rate of 15 miles per hour. Find the apparent velocity and direction of motion of the second ship to an observer on the first ship.

4. A ship steams due west at the rate of 15 miles per hour relative to the current which is flowing at the rate of 6 miles per hour due south. What is the velocity relative to the ship of a train going due north at the rate of 30 miles per hour?

5. In a tunnel, drops of water which are falling from the roof are noticed to pass the carriage window of a train in a direction making an angle $\tan^{-1} \frac{1}{2}$ with the horizon, and they are known to have a velocity of 24 feet per second. Neglecting the resistance of the air, find the velocity of the train.

6. A steamer is going due west at 14 miles per hour, and the wind appears from the drift of the clouds to be blowing at 7 miles per hour from the north-west. Find its actual velocity and make a geometrical construction for its direction.

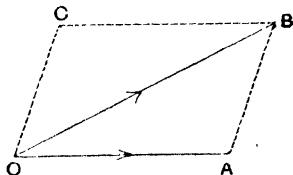
7. Two trains, each 200 feet long, are moving towards each other on parallel lines with velocities of 20 and 30 miles per hour respectively. Find the time that elapses from the instant when they first meet until they have cleared each other.

8. A ship is sailing north-east with a velocity of 10 miles per hour and to a passenger on board the wind appears to blow from the north with a velocity of $10\sqrt{2}$ miles per hour. Find the true velocity and direction of the wind.

9. A motor-car is travelling at 15 miles per hour towards the south-west, and the wind appears to a passenger in it to blow from the west with a velocity of 12 miles per hour. Find, graphically or otherwise, the true velocity and direction of the wind.

213. Change of Velocity. Suppose a point at any instant to be moving with a velocity represented by OA , and that at some subsequent time its velocity is represented by OB .

Join AB , and complete the parallelogram $OACB$.



Then velocities represented by OA and OC are equivalent to the velocity OB . Hence the velocity OC is the velocity which must be compounded with OA to produce the velocity OB . The velocity OC is therefore the change of velocity in the given time.

Thus the change of velocity is not, in general, the difference in magnitude between the magnitudes of the two velocities, but is that velocity which compounded with the original velocity gives the final velocity.

The change of velocity is not constant unless it is constant both in magnitude and direction.

Ex. 1. A point is moving with a velocity of 10 feet per second, and at a subsequent instant it is moving at the same rate in a direction inclined at 30° to the former direction; find the change of velocity.

On drawing the figure, as in the last article, we have $OA = OB = 10$, and the angle $AOB = 30^\circ$.

Since $OA = OB$, we have $\angle OAB = 75^\circ$, and therefore $\angle AOC = 105^\circ$.

The change in the velocity, i.e., OC ,

$$= AB = \sqrt{10^2 + 10^2 - 2 \cdot 10 \cdot 10 \cos 30^\circ} = 5\sqrt{8 - 4\sqrt{3}} = 5(\sqrt{6} - \sqrt{2}),$$

and is in a direction inclined at 105° to the original direction of motion.

Ex. 2. A point is moving with a velocity of 5 feet per second, and at a subsequent instant it is moving at the same rate in a direction inclined at 60° to its former direction; shew that the change of velocity is 5 feet per second at 120° with its original direction.

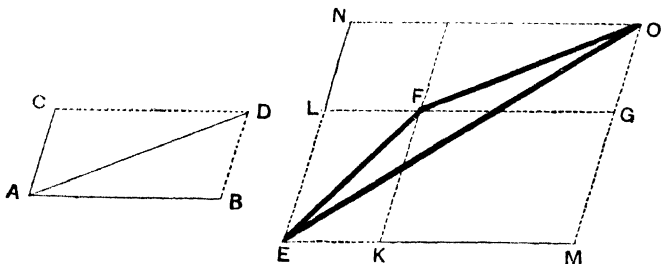
Ex. 3. A point is moving eastward with a velocity of 20 feet per second, and one hour afterwards it is moving north-east with the same speed; shew that the change of velocity is $20\sqrt{2 - \sqrt{2}}$, i.e., 15.31 feet per second towards N.N.W.

Ex. 4. A point is describing with uniform speed a circle, of radius 7 yards, in 11 seconds, starting from the end of a fixed diameter; shew that the change in its velocity after it has described one-sixth of the circumference is 12 feet per second at 120° with the original direction.

214. Theorem. Parallelogram of Accelerations.

If a moving point have simultaneously two accelerations represented in magnitude and direction by two sides of a parallelogram drawn from a point, they are equivalent to an acceleration represented by the diagonal of the parallelogram passing through that angular point.

Let the accelerations be represented by the sides AB and AC of the parallelogram $ABDC$, i.e. let AB and AC represent the velocities added to the velocity of the point in a



unit of time. On the same scale let EF represent the velocity which the particle has at any instant. Draw the parallelogram $EKFL$ having its sides parallel to AB and AC ; produce EK to M , and EL to N , so that KM and LN are equal to AB and AC respectively. Complete the parallelograms as in the above figure.

Then the velocity EF is equivalent to velocities EK and EL . But in the unit of time the velocities KM and LN are the changes of velocity.

Therefore at the end of a unit of time the component velocities are equivalent to EM and EN , which are equivalent to EO , and this latter velocity is equivalent to velocities EF and FO . (Art. 210.)

Hence in the unit of time FO is the change of velocity of the moving point, i.e. FO is the resultant acceleration of the point.

But FO is equal and parallel to AD .

Hence AD represents the acceleration which is equivalent to the accelerations AB and AC , i.e. AD is the resultant of the accelerations AB and AC .

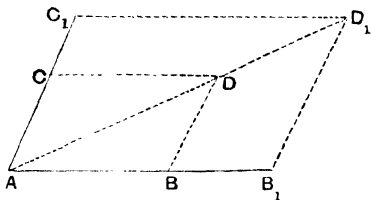
It follows from this article that accelerations are resolved and compounded in the same way as velocities, and propositions similar to those of Arts. 208–211 will be true when we substitute “acceleration” for “velocity.”

Velocities and accelerations, and also forces (Art. 21), are examples of an important class of physical quantities which are called Vector quantities. The characteristic of a Vector quantity is that it has direction as well as magnitude, and is thus fitly represented by a straight line; in all cases vector quantities are compounded by the parallelogrammic law.

In the language of Vectors Arts. 207 and 214 are examples of the Addition of Vectors, and it would be said that the addition of the vectors AB and BD (or AC) gives the vector AD .

In contradistinction to Vectors, quantities which only possess magnitude, and not direction, are called Scalars. Kinetic Energy is an example of a physical quantity which is a Scalar; other examples are a ton of coal, a sum of money, etc. Scalar quantities are compounded by Simple Addition.

215. Parallelogram of Forces. We have shewn in the last article that if a particle of mass m have accelerations f_1 and f_2 represented in magnitude and direction by lines AB and AC , then its resultant acceleration f_3 is represented in magnitude and direction by AD , the diagonal of the parallelogram of which AB and AC are adjacent sides.



Since the particle has an acceleration f_1 in the direction AB there must be a force $P_1 (= mf_1)$ in that direction, and similarly a force $P_2 (= mf_2)$ in the direction AC . Let AB_1 and AC_1 represent these forces in magnitude and direction. Complete the parallelogram $AB_1D_1C_1$. Then since the forces in the directions AB_1 and AC_1 are proportional to the accelerations in those directions,

$$\therefore AB_1 : AB :: B_1D_1 : BD.$$

Hence, by simple geometry, we have A , D and D_1 in a straight line, and

$$\begin{aligned} AB_1 : AB &:: AC_1 : AC \\ &:: B_1D_1 : BD. \end{aligned}$$

Hence AD_1 represents the force which produces the acceleration represented by AD , and hence is the force which is equivalent to the forces represented by AB_1 and AC_1 .

Hence we infer the truth of the Parallelogram of Forces (Art. 21).

216. Physical Independence of Forces. The latter part of the Second Law of Motion states that the change of motion produced by a force is in the direction in which the force acts.

Suppose we have a particle in motion in the direction AB and a force acting on it in the direction AC ; then the law states that the velocity in the direction AB is unchanged, and that the only change of velocity is in the direction AC ; so that to find the real velocity of the particle at the end of a unit of time, we must compound its velocity in the direction AB with the velocity generated in that unit of time by the force in the direction AC . The same reasoning would hold if we had a second force acting on the particle in some other direction, and so for any system of forces. Hence if a set of forces act on a particle at rest, or in motion, their combined effect is found by considering the effect of each force on the particle *just as if the other forces did not exist, and as if the particle were at rest*, and then compounding these effects. This principle is often referred to as that of the *Physical Independence of Forces*.

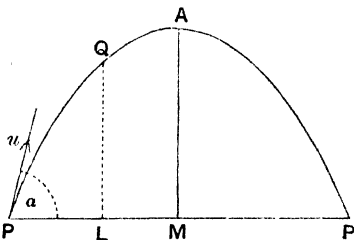
As an illustration of this principle consider the motion of a ball allowed to fall from the hand of a passenger in a train which is travelling rapidly. It will be found to hit the floor of the carriage at exactly the same spot as it would have done if the carriage had been at rest. This shews that the ball must have continued to move forward with the same velocity that the train had, or, in other words, the weight of the body only altered the motion in the vertical direction, and had no influence on the horizontal velocity of the particle.

Again, a circus-rider, who wishes to jump through a hoop, springs in a vertical direction from the horse's back; his horizontal velocity is the same as that of the horse and remains unaltered; he therefore alights on the horse's back at the spot from which he started.

Projectiles.

217. By the use of the principle of the last article we can determine the motion of a particle which is thrown into the air, not necessarily in a vertical line, but in any direction whatever.

Let the particle be projected from a point P with velocity u in a direction making an angle α with the horizon; also let PAP' be the path of the particle, A being the highest point, and P' the point where the path again meets the horizontal plane through P . The distance PP' is called the range on the horizontal plane through P .



Now the weight of the body only has effect on the motion of the body in the vertical direction; it therefore has no effect on the velocity of the body in the horizontal direction, and this horizontal velocity therefore remains unaltered (since the resistance of the air is disregarded).

The horizontal and vertical components of the initial velocity of the particle are $u \cos \alpha$ and $u \sin \alpha$ respectively.

The horizontal velocity is, therefore, throughout the motion equal to $u \cos \alpha$.

In the vertical direction the initial velocity is $u \sin \alpha$ and the acceleration is $-g$, [for the acceleration due to gravity is g vertically *downwards*, and we are measuring our positive direction *upwards*]. Hence the vertical motion is the same as that of a particle projected vertically upwards with velocity $u \sin \alpha$, and moving with acceleration $-g$.

The resultant motion of the particle is the same as that of a particle projected with a vertical velocity $u \sin \alpha$ inside a vertical tube of small bore, whilst the tube moves in a horizontal direction with velocity $u \cos \alpha$.

The **time of flight**, t , *i.e.* the time the projectile is in the air, is therefore twice the time in which a vertical velocity $u \sin \alpha$ is destroyed by g , *i.e.* $t = 2 \frac{u \sin \alpha}{g}$.

The **greatest height** attained is the space in which a velocity $u \sin \alpha$ is destroyed by gravity, and hence, as in Art. 165, is equal to

$$\frac{u^2 \sin^2 \alpha}{2g}.$$

Also the horizontal range

$$PP' = u \cos \alpha \times t = 2 \frac{u^2 \sin \alpha \cos \alpha}{g}.$$

The curve which is described by the particle is called a **Parabola**.

Ex. 1. A cannon-ball is projected horizontally from the top of a tower, 49 feet high, with a velocity of 200 feet per second. Find

- (1) the time of flight,
- (2) the distance from the foot of the tower of the point at which it hits the ground, and
- (3) its velocity when it hits the ground.

(1) The initial vertical velocity of the ball is zero, and hence t , the time of flight, is the time in which a body, falling freely under gravity, would describe 49 feet.

$$\text{Hence, by Art. 158,} \quad 49 = \frac{1}{2}g \cdot t^2 = 16t^2.$$

$$\therefore t = \frac{7}{4} \text{ seconds.}$$

(2) During this time the horizontal velocity remains constant, and therefore the required distance from the foot of the tower

$$= 200 \times \frac{7}{4} = 350 \text{ feet.}$$

(3) The vertical velocity at the end of $\frac{7}{4}$ seconds $= \frac{7}{4} \times 32 = 56$ feet per second, and the horizontal velocity is 200 feet per second.

Hence the required velocity $= \sqrt{200^2 + 56^2} = 8\sqrt{674} = 207.7$ feet nearly, at an angle θ to the ground where $\tan \theta = \frac{56}{200} = .28$, so that $\theta = 15^\circ 38\frac{1}{2}'$.

Ex. 2. From the top of a cliff, 80 feet high, a stone is thrown so that it starts with a velocity of 128 feet per second, at an angle of 30° with the horizontal; find where it hits the ground at the bottom of the cliff.

The initial vertical velocity is $128 \sin 30^\circ$, or 64, feet per second, and the initial horizontal velocity is $128 \cos 30^\circ$, or $64\sqrt{3}$, feet per second.

Let t be the time that elapses before the stone hits the ground.

Then t is the time in which a stone, projected with vertical velocity 64 and moving with acceleration $-g$, describes a distance -80 feet.

$$\therefore -80 = 64t - \frac{1}{2}gt^2.$$

Hence $t = 5$ seconds.

During this time the horizontal velocity remains unaltered, and hence the distance of the point, where the stone hits the ground, from the foot of the cliff $= 320\sqrt{3} = \text{about } 554$ feet.

Ex. 3. A bullet is projected, with a velocity of 640 feet per second, at an angle of 30° with the horizontal; find (1) the greatest height attained, and (2) the range on a horizontal plane and the time of flight.

The initial horizontal velocity

$$= 640 \cos 30^\circ = 640 \times \frac{\sqrt{3}}{2} = 320\sqrt{3} \text{ feet per second.}$$

The initial vertical velocity $= 640 \sin 30^\circ = 320$ feet per second.

(1) If h be the greatest height attained, then h is the distance through which a particle, starting with velocity 320 and moving with acceleration $-g$, goes before it comes to rest.

$$\therefore 0 = 320^2 - 2gh;$$

$$\therefore h = \frac{320^2}{2 \times 32} = 1600 \text{ feet.}$$

(2) If t be the time of flight, the vertical distance described in time t is zero.

$$\therefore 0 = 320t - \frac{1}{2}gt^2;$$

$$\therefore t = \frac{640}{g} = 20 \text{ seconds.}$$

The horizontal range = the distance described in 20 seconds by a particle moving with a constant velocity of $320\sqrt{3}$ ft. per sec.

$$= 20 \times 320\sqrt{3} = 11,085 \text{ feet approximately.}$$

EXAMPLES. XXXVII

1. A particle is projected at an angle α to the horizon with a velocity of u feet per second; find the greatest height attained, the time of flight, and the range on a horizontal plane, when

$$(1) \quad u = 64, \quad \alpha = 30^\circ;$$

and

$$(2) \quad u = 80, \quad \alpha = 60^\circ.$$

2. A projectile is fired horizontally from a height of 9 feet from the ground, and reaches the ground at a horizontal distance of 1000 feet. Find its initial velocity.

3. A stone is dropped from a height of 9 feet above the floor of a railway carriage which is travelling at the rate of 30 miles per hour. Find the velocity and direction of the particle in space at the instant when it meets the floor of the carriage.

4. A ship is moving with a velocity of 16 feet per second, and a body is allowed to fall from the top of its mast, which is 144 feet high; find the velocity and direction of motion of the body, (1) at the end of two seconds, and (2) when it hits the deck.

5. A particle is projected horizontally from the top of a tower at the rate of 10 miles per hour and falls under the action of gravity. Assuming that no other forces are acting draw a figure to represent its position at the end of $1, 1\frac{1}{2}, 2\frac{1}{2}$, and 3 seconds.

6. A balloon is carried along at a height of 100 feet from the ground at the rate of 40 miles per hour and a stone is dropped from it; find

the time that elapses before it reaches the ground, and the distance from the point where it reaches the ground to the point vertically below the point where it left the balloon.

7. A stone is thrown horizontally, with velocity $\sqrt{2gh}$, from the top of a tower of height h . Find where it will strike the level ground through the foot of the tower. What will be its striking velocity?

8. A shot is fired from a gun on the top of a cliff, 400 feet high, with a velocity of 768 feet per second, at an elevation of 30° . Find the horizontal distance from the vertical line through the gun of the point where the shot strikes the water.

9. From the top of a vertical tower, whose height is $\frac{1}{2}g$ feet, a particle is projected, the vertical and horizontal components of its initial velocity being $6g$ and $8g$ respectively; find the time of flight, and the distance from the foot of the tower of the point at which it strikes the ground.

10. On the moon there seems to be no atmosphere, and gravity there is about one-sixth of that on the earth. What space of country would be commanded by the guns of a lunar fort able to project shot with a velocity of 1600 feet per second?

11. If the range of a projectile on a horizontal plane is 100 yards, and the time of flight is 5 seconds, shew that the velocity of projection is 100 feet per second.

Uniform motion in a circle

218. We have learnt from the First Law of Motion that every particle, once in motion, will, unless it be prevented from so doing, continue to move in a straight line with unchanged velocity. Hence a particle will not describe a circle, or any *curved* path, unless it be compelled to do so. When a particle is describing a circle, in such a manner that the magnitude of its velocity is constant, the direction of its velocity is continually changing. There is therefore a change in its velocity (Art. 213) and so it moves with an acceleration.

219. If a particle describes a circle of radius r so that the magnitude of its velocity is v , it can be proved (*Elements of Dynamics*, Art. 135) that its acceleration f is always equal to $\frac{v^2}{r}$ and that it is always in a direction towards the centre of the circle. We shall assume this result.

Hence
$$f = \frac{v^2}{r}.$$

Now, by the Second Law of Motion, wherever there is acceleration, there must be force to produce it. Also, by Art. 173, we know that the force P required to produce in a mass m an acceleration f is given by $P = mf$.

Hence, if a particle describe with velocity v a circle of radius r , it must be acted on by a force P directed towards the centre of the circle, such that

$$P = mf = m \frac{v^2}{r}.$$

220. The force spoken of in the preceding article exhibits itself in various forms.

As a simple example consider the case of a particle tied to one end of a string, the other end of which is attached to a point of a smooth horizontal table. Let the string be stretched out and laid flat on the table, and let the particle be struck so as to start moving on the table in a direction at right angles to the string. It will describe a circle about the fixed end of the string as centre. In this case the tension of the string supplies the force requisite to make the particle move in a circle.

Ex. A particle, of mass 3 lbs., moves on a smooth table with a velocity of 4 feet per second, being attached to a fixed point on the table by a string of length 5 feet; find the tension of the string.

Here $v = 4$, and $r = 5$.

Hence, by the last article, the acceleration of the particle is towards the fixed point and equals $\frac{v^2}{r}$, i.e. $\frac{16}{5}$ ft.-sec. units.

Hence the tension of the string

$$\begin{aligned} &= m \frac{v^2}{r} = 3 \times \frac{16}{5} = \frac{48}{5} \text{ poundals} \\ &= \text{wt. of } \frac{48}{5 \times 32}, \text{ i.e. } \frac{3}{10}, \text{ of a pound.} \end{aligned}$$

If the string were so weak that it could not exert this tension it would break; the particle would then proceed to describe a straight line on the table.

221. In the case of a locomotive engine moving on horizontal rails round a curve, the required force is provided by the pressure between the rails and the flanges of the wheels. If the rails were horizontal and at the same level, and if there were no flanges to the wheels, the engine would not describe a curved portion of the line but would leave the rails.

Ex. A locomotive engine, of mass 10 tons, moves on a curve, whose radius is 600 feet, with a velocity of 15 miles per hour; what force must be exerted by the rails?

15 miles per hour = 22 feet per second.

$$\text{Hence } \frac{v^2}{r} = \frac{22^2}{600}.$$

Also the mass of the engine = 2240×10 lbs.

$$\begin{aligned} \text{Hence the force required} &= m \frac{v^2}{r} \\ &= 2240 \times 10 \times \frac{22^2}{600} \text{ poundals} \\ &= \frac{10}{32} \times \frac{22^2}{600} \text{ ton weight} \\ &= \frac{121}{480}, \text{ i.e. nearly } \frac{1}{4}, \text{ ton wt.} \end{aligned}$$

EXAMPLES. XXXVIII

1. A body, of mass 20 lbs., describes a circle of radius 10 ft. with a velocity of 15 ft. per second. Find the force required to make it do so.

2. What must be the force that acts towards the centre of a circle, whose radius is 5 ft., to make a body of mass 10 lbs. describe the circle with a velocity of 20 ft. per second?

3. With what velocity must a mass of 10 grammes revolve horizontally at the end of a string, half a metre long, to cause the same tension in the string as would be caused by a mass of one gramme hanging vertically at the end of a similar string?

4. A string, 5 ft. long, can just support a weight of 16 lbs. and has one end tied to a point on a smooth horizontal table. At the other end is tied a mass of 10 lbs. What is the greatest velocity with which the mass can be projected on the table so that the string may not break?

5. An engine, of mass 9 tons, passes round a curve, half a mile in radius, with a velocity of 35 miles per hour. What force tending towards the centre of the curve must be exerted by the rails?

6. If in the previous question the mass of the engine be 12 tons, its velocity 60 miles per hour, and the radius of the curve be 400 yds., what is the required force?

HYDROSTATICS

CHAPTER XVII

FLUID PRESSURE

222. IN Statics we have considered the equilibrium of rigid bodies only, and we have defined a rigid body as one the particles of which always retain the same position with respect to one another. A rigid body possesses therefore a definite *size* and *shape*. We have pointed out that there are no such bodies in Nature, but that there are good approximations thereto.

In Hydrostatics we consider the equilibrium of such bodies as water, oils, and gases. The common distinguishing property of such bodies is the ease with which their portions can be separated from one another.

If a very thin lamina be pushed *edgeways* through water the resistance to its motion is very small, so that the force of the nature of friction, *i.e.* along the surface of the lamina, must be very small. There are no fluids in which this force quite vanishes, but throughout this book we shall assume that no such force exists in the fluids we have to deal with. Such a hypothetical fluid is called a perfect fluid, the definition of which may be formally given as follows:

223. Perfect Fluid. Def. A perfect fluid is a substance such that its shape can be altered by any tangential force however small, if applied long enough, of which portions can be easily separated from the rest of the mass, and between different portions of which there is no tangential, *i.e.* rubbing, force of the nature of friction. The difference between a perfect fluid and a fluid such as water is chiefly seen in the case of the motion of the water.

For example, if we set water revolving in a cup, the frictional resistances between the water and the cup and between different portions of the water soon reduce it to

rest. When water is at rest it practically is equivalent to a perfect fluid.

224. Fluids are again subdivided into two classes, viz. **Liquids** and **Gases**.

Liquids are substances such as water and oils. They are almost entirely incompressible. An incompressible body is one whose *total volume*, *i.e.* the space it occupies, cannot be increased or diminished by the application of any force, however great, although any force, however small, would change its *shape*. All liquids are really compressible under very great pressure but only to a very slight degree. For example, a pressure equal to about 200 times that of the atmosphere will only reduce the volume of a quantity of water by a one-hundredth part. This compressibility we shall neglect, and therefore define liquids as those fluids which are incompressible.

Gases, on the other hand, are fluids which can easily be made to change their total volume, *i.e.* which are, more or less easily, compressible.

If a child's air-ball be placed under the receiver of an air-pump from which the air has been excluded it will increase very much in size. If the skin of the air-ball be broken the air will expand and fill the receiver whatever be the size of the latter.

225. The definitions of a liquid and gas may be formally stated as follows:

A perfect liquid is a fluid which is absolutely incompressible.

A gas is a fluid such that a finite quantity of it will, if the pressure to which it is subjected be sufficiently diminished, expand so as to fill any space however great.

226. The difference between a rigid body, a liquid, and a gas may be thus expressed:

A perfectly rigid body has a definite size and a definite shape.

A perfect liquid has a definite size but no definite shape.

A perfect gas has no definite size and no definite shape.

227. *Viscous fluids.* No fluids are perfect. Many fluids, such as treacle, honey, and tar, offer a considerable resistance

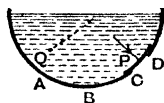
to forces which tend to alter their shape. Such fluids, in which the tangential or rubbing action between layers in contact cannot be neglected, are called viscous fluids.

228. Pressure at a point. Suppose a hole to be made in the side of a vessel containing fluid, and that this hole is covered by a plate which exactly fits the hole. The plate will not remain at rest unless it be kept at rest by the application of some force; in other words the fluid exerts a force on the plate.

Also the fluid can, by definition, only exert a force *perpendicular* to each element of area of this plate.

If the force exerted by the fluid on each equal element of area of the plate be the same, the pressure *at* any point of the plate is the force which the fluid exerts on the unit of area surrounding that point.

If, however, the force exerted by the fluid on each equal element of the area of the plate be not the same, as in the case of the plate *CD*, the pressure at any point *P* of this plate is that force which the fluid would exert on a unit of area at *P*, if on this unit of area the pressure were uniform and the same as it is on an indefinitely small area at *P*.



The pressure at any point within the fluid, such as *Q*, is thus obtained. Suppose an indefinitely small rigid plate placed at *Q* so as to contain *Q* and let its area be a square feet. Imagine all the fluid on one side of this plate removed and that, to keep the plate at rest, a force of *X* lbs. wt. must be applied to it. The pressure *at* the point *Q* is then $\frac{X}{a}$ lbs. wt. per square foot.

229. The theoretical unit of pressure is, in the foot-pound system of units, one poundal per square foot. In the c.g.s. system the corresponding unit is one dyne per square centimetre.

In practice the pressure at any point of a fluid is not usually expressed in *poundals* per square foot but in *lbs. wt.* per square *inch*. The former measure is however the best for theoretical calculation and may be easily converted into the latter.

Similarly, in the c.g.s. system, the practical measure of a pressure is in *grammes weight* per square centimetre.

It is sometimes convenient to convert a pressure expressed in the

foot-pound system into the c.g.s. system. As in Arts. 151 and 169 we have

$$1 \text{ inch} = 2.54 \text{ cms.}; \quad 1 \text{ cm.} = .3937 \text{ inch.}$$

$$1 \text{ lb.} = 453.6 \text{ grammes}; \quad \text{and } 1 \text{ gramme} = .002204 \text{ lb.}$$

Hence a pressure of 1 lb. wt. per sq. inch

$$= \text{a pressure of } 453.6 \text{ grammes wt. per } (2.54)^2 \text{ sq. cms.}$$

$$= \text{a pressure of } \frac{453.6}{(2.54)^2} \text{ grammes wt. per sq. cm.}$$

$$= \text{a pressure of } 70.31 \text{ grammes wt. per sq. cm.}$$

Conversely, a pressure of 1 gramme wt. per sq. cm.

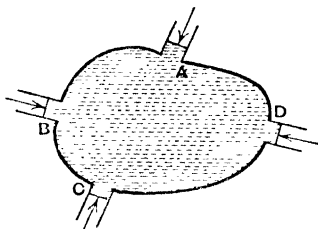
$$= \text{a pressure of } .0142 \text{ lb. wt. per sq. inch.}$$

230. Transmission of fluid pressure. *If any pressure be applied to the surface of a fluid it is transmitted equally to all parts of the fluid.*

This proposition may be proved experimentally as follows:

Let fluid be contained in a vessel of any shape and in the vessel let there be holes $A, B, C, D, \dots$ of various sizes, which are stopped by tight-fitting pistons to which forces can be applied.

Let the areas of these pistons be $a, b, c, d, \dots$ square feet, and let the pistons be kept in equilibrium by forces applied to them.



If an additional force $p \cdot a$ be applied to A [*i.e.* an additional pressure of p lbs. wt. per unit of area of A] it is found that an additional force of $p \cdot b$ lbs. wt. must be applied to B , one of $p \cdot c$ lbs. wt. to C , one of $p \cdot d$ lbs. wt. to D , and so on, whatever be the number of pistons. Hence an additional pressure of p , per unit of area, applied to A causes an additional pressure of p , per unit of area, on B , and of the same additional pressure per unit of area on each of the other pistons $C, D, \dots$

Hence the proposition is proved.

231. *The pressure at any point of a fluid at rest is the same in all directions.*

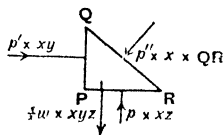
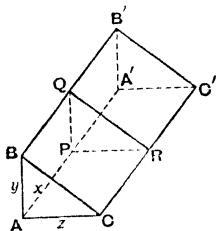
This may be proved experimentally by a modification of the experiment of the last article.

For suppose any one of the pistons, D , to be so arranged that it may be turned into any other position, *i.e.* so that its plane may be made parallel to the planes of either A , B , or C , or be made to take any other position whatever. *It is found* that the application of an additional pressure at A , of p per unit of area, produces the same additional pressure, of p per unit of area, at D whatever be the position that D is made to occupy.

***232.** The foregoing proposition may be deduced from the fundamental fact that the pressure of a fluid is always perpendicular to any surface with which it is in contact.

Consider any portion of fluid in the shape of a triangular prism having its base $ACC'A'$ horizontal, and its face $ABB'A'$ and its two triangular ends ABC and $A'B'C'$ all vertical.

Let the length AA' , the breadth AC , and the height AB be all very small and let P , Q , and R be respectively the middle points of AA' , BB' , and CC' .



Let the lengths of AA' , AB , and AC be x , y , and z respectively.

Since the edges x and z are very small the pressure on the face $AA'C'C$ may be considered to be uniform, so that, if p be the pressure on it per unit of area, the force exerted by the fluid on it is $p \times xz$ and acts at the middle point of PR .

So if p' and p'' be the pressures per unit of area on $AA'B'B$ and $BCC'B'$ respectively, the forces on these areas are $p' \times xy$ and $p'' \times x \cdot QR$ acting at the middle points of PQ and QR respectively.

If w poundals be the weight of the fluid per unit of volume, then, since the volume of the prism is $AA' \times \text{area } ABC$, *i.e.* $x \times \frac{1}{2}yz$, the weight of the fluid prism is $w \times \frac{1}{2}xyz$ and acts vertically through the centre of gravity of the triangle PQR .

This weight and the three forces exerted on the face must be a system of forces in equilibrium; for otherwise the prism would move.

Hence, resolving the forces horizontally, we have

$$p' \cdot xy = p'' \times x \cdot QR \times \cos(90^\circ - R) = p'' \times x \cdot QR \times \sin R \quad [\text{App. I, Art. 8}]$$

$$= p'' \times x \cdot PQ = p'' \cdot xy,$$

so that

$$p' = p'' \dots \dots \dots (1).$$

Again, resolving vertically, we have

$$\begin{aligned} p \times xz - w \times \frac{1}{2}xyz &= p'' \times x \cdot QR \times \sin(90^\circ - R) \\ &= p'' \times x \cdot QR \times \cos R = p'' \times x \cdot PR = p'' \times xz, \\ \therefore p - p'' &= w \cdot \frac{1}{2}y \dots\dots\dots(2). \end{aligned}$$

Now let the sides of the prism be taken indefinitely small (in which case p , p' , and p'' are the pressures at the point P in directions perpendicular respectively to PR , PQ , and QR). The quantity $w \cdot \frac{1}{2}y$ now becomes indefinitely small and therefore negligible.

The equation (2) then becomes

$$p = p'',$$

so that

$$p = p' = p''.$$

Now the direction of BC is any that we may choose, so that it follows that the pressure of the fluid at P is the same in all directions.

233. Bramah's or the Hydrostatic Press.

Bramah's press affords a simple example of the transmission of fluid pressures.

In its simplest shape it consists of two cylinders $ABCD$ and $EFGH$ both containing water, the two cylinders being connected by a tube CG . The section of one cylinder is very much greater than that of the other.

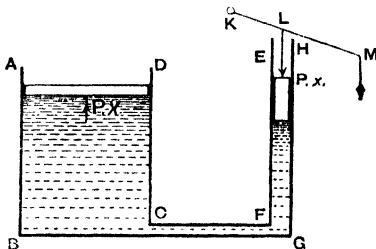
In each cylinder is a closely fitting water-tight piston, the areas of the sections of which are X and x .

To the smaller piston a pressure equal to P lbs. wt. per unit of its area is applied, so that the total force applied to it is $P \cdot x$ lbs. wt.

By Art. 230 a pressure of P lbs. wt. per unit of area will be transmitted throughout a fluid, so that the force exerted by the fluid on the piston in the larger cylinder will be $P \cdot X$ lbs. wt.

This latter force would support on the upper surface of the piston a body whose weight is $P \cdot X$ lbs.

$$\text{Hence } \frac{\text{weight of the body supported}}{\text{force applied}} = \frac{P \cdot X}{P \cdot x} = \frac{X}{x},$$



so that the force applied becomes multiplied in the ratio of X to x , *i.e.* in the ratio of the areas of the sections of the two cylinders.

In the above investigation the weights of the two pistons have been neglected, and also the difference between the levels of the fluid in the two cylinders.

The pressure is usually applied to the smaller piston by means of a lever KLM , which can turn freely about its end K which is fixed. At M the effort is applied, and the point L is attached to the smaller piston by a rigid rod.

Theoretically, we could by making the small piston small enough and the large piston large enough, multiply to any extent the force applied. Practically, this multiplication is limited by the fact that the sides of the vessel would have to be immensely strong to support the thrusts that would be put upon them.

234. Ex. *If the area of the small piston in a Bramah's Press be $\frac{1}{3}$ sq. inch and that of the large piston be 2 square feet, what weight would be supported by the application of 20 lbs. wt. to the smaller piston?*

The pressure at each point of the fluid in contact with the small piston is $20 \div \frac{1}{3}$, *i.e.* 60 lbs. wt. per square inch.

This pressure is (by Art. 230) applied to each square inch of the larger piston whose area is 288 sq. ins.

Hence the total force exerted on the large piston is 288×60 , *i.e.* 17,280 lbs. wt., *i.e.* $7\frac{5}{7}$ tons wt.

A weight of $7\frac{5}{7}$ tons would therefore be supported by the larger piston.

235. Bramah's Press forms a good example of the Principle of Work as enunciated in Art. 139.

For, since the decrease in the volume of the water in the small cylinder is equal to the increase of the water in the large cylinder, it follows that

$$X \cdot Y = x \cdot y,$$

where Y and y are the respective distances through which the large and small pistons move.

Hence
$$\frac{X}{x} = \frac{y}{Y}.$$

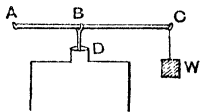
$$\therefore \frac{\text{force exerted by the large piston}}{\text{force exerted by the small piston}} = \frac{X}{x} = \frac{y}{Y}.$$

$$\begin{aligned} \therefore \text{force exerted by the large piston} \times Y \\ = \text{force exerted by the small piston} \times y, \end{aligned}$$

i.e. the work done by the large piston is the same as that done on the small piston. Hence the Principle of Work holds.

236. Safety Valve. The safety valve affords another example of the pressure exerted by fluids. In the case of a boiler with steam inside it, the pressure of the steam might often become too great for the strength of the boiler, and there would be danger of its bursting. The use of the safety valve is to allow the steam to escape when the pressure is greater than what is considered to be safe.

In one of its forms it consists of a circular hole D in the side of the boiler into which there fits a plug. This plug is attached at B to an arm ABC , one end of which, A , is jointed to a fixed part of the boiler. The arm ABC can turn about A and at the other end C can be attached whatever weights are desired.



It is clear that the pressure of the steam and the weight at C tend to turn the arm in different directions. When the moment of the pressure of the steam about A is greater than the moment of the weight at C , the plug D will rise and allow steam to escape, thus reducing the pressure.

In other valves there is no lever ABC and the plug is replaced by a circular valve, which is weighted and which can turn about a point in its circumference.

Ex. The arms of the lever of a safety valve are 1 inch and 18 inches, and at the end of the longer arm is hung a weight of 20 lbs. If the area of the valve be $1\frac{1}{2}$ square inches, what is the maximum pressure of the steam which is allowed?

If p lbs. wt. per square inch be the required pressure, the total force exerted on the valve by the steam $= p \times \frac{3}{2}$ lbs. wt.

When the valve is just going to rise, the two forces $\frac{3p}{2}$ and 20 lbs. wt. balance at the ends of arms 1 and 18 inches.

$$\text{Hence} \quad \frac{3p}{2} \times 1 = 20 \times 18.$$

$$\therefore p = 240 \text{ lbs. wt.}$$

EXAMPLES. XXXIX

1. In a Bramah's Press the diameters of the large and small pistons are respectively $2\frac{1}{2}$ decimetres and 2 centimetres; a kilogram is placed on the top of the small piston; find the mass which it will support on the large piston.

2. In a Bramah's Press the area of the large piston is 100 square inches and that of the small one is $\frac{1}{4}$ square inch; find the force that must be applied to the latter so that the former may lift 1 ton.

3. A water cistern, which is full of water and closed, can just bear a pressure of 1500 lbs. wt. per square foot without bursting.

If a pipe whose section is $\frac{1}{4}$ square inch communicate with it and be filled with water, find the greatest weight that can safely be placed on a piston fitting this pipe.

4. In a Bramah's Press if a pressure of 1 ton wt. be produced by an effort of 5 lbs. wt. and the diameters of the pistons be in the ratio of 8 to 1, find the ratio of the lengths of the arms of the lever employed to work the piston.

5. In a hydraulic press the radii of the cylinders are 3 inches and 6 feet respectively. The effort is applied at the end of a lever whose length is 2 feet, the piston being attached at a distance of 2 inches from the fulcrum. If a body weighing 10 tons be placed upon the large piston, find the power that must be applied to the lever.

If the materials of the press will only bear a pressure of 150 lbs. wt. to the square inch, find what is the greatest weight that can be lifted.

6. A vessel full of water is fitted with a tight cork. How is it that a slight blow on the cork may be sufficient to break the vessel?

7. The arms of the lever of a safety valve are of lengths 2 inches and 2 feet, and at the end of the longer arm is suspended a weight of 12 lbs. If the area of the valve be 1 square inch, what is the pressure of the steam in the boiler when the valve is raised?

8. Find the pressure of steam in a boiler when it is just sufficient to raise a circular safety valve which has a diameter of $\frac{1}{8}$ inch and is loaded so as to weigh $\frac{1}{2}$ lb.

9. The weight of the safety valve of a steam boiler is 16 lbs. and its section is $\frac{1}{6}$ of a square inch. Find the pressure of the steam in the boiler that will just lift the safety valve.

CHAPTER XVIII

DENSITY AND SPECIFIC GRAVITY

237. Density. Def. The density of a homogeneous body is the mass of a unit volume of the body.

The mass of a cubic foot of water is found to be about 1000 ozs., *i.e.* $62\frac{1}{2}$ lbs. Hence the density of water is $62\frac{1}{2}$ lbs. per cubic foot.

A gramme is the mass of the water at 4°C . which would fill a cubic centimetre. Hence the density of water at 4°C . is one gramme per cubic centimetre.

The reason why a certain temperature is taken when we define a gramme is that the volume of a given mass of water alters with the temperature of the water. If we take a given mass (say 1 lb.) of water and cool it gradually from the boiling point 100°C . [*i.e.* 212°F .], it is found to occupy less and less space until the temperature is reduced to 4°C . [about $39\cdot2^{\circ}\text{F}$.]. If the temperature be continually lowered still further the volume occupied by the pound of water is now found to *increase* until the water arrives at its freezing point. Hence the pound of water occupies less space at 4°C . than at any other temperature, *i.e.* there is more water in a given volume at 4°C . than at any other temperature, *i.e.* the density is greatest at 4°C .

The mass of a cubic foot of mercury is found to be 13·596 times that of a cubic foot of water. Hence the density of mercury is nearly $13\cdot596 \times 62\frac{1}{2}$ lbs. per cubic foot.

If we use centimetre-gramme units the density of mercury is 13·596 grammes per cubic centimetre.

It is sometimes convenient to be able to convert densities expressed in the foot-pound system into the c.g.s. system, and conversely.

As in Arts. 151 and 169 we have

$$1 \text{ foot} = 30\cdot48 \text{ centimetres}; \quad 1 \text{ cm.} = \cdot0328 \text{ ft.}$$

$$1 \text{ lb.} = 453\cdot6 \text{ grammes}; \quad 1 \text{ gramme} = \cdot002204 \text{ lb.}$$

Hence a density of 1 lb. per cubic foot

$$= \text{a density of } 453\cdot6 \text{ grammes per } (30\cdot48)^3 \text{ cubic cms.}$$

$$= \text{a density of } \frac{453\cdot6}{(30\cdot48)^3} \text{ gramme per cub. cm.}$$

$$= \cdot01602 \text{ gramme per cub. cm. approx.}$$

So a density of 1 gramme per cub. cm.

= a density of $\cdot 002204$ lb. per $(\cdot 0328)^3$ cub. ft.

= a density of $\frac{\cdot 002204}{(\cdot 0328)^3}$ lbs. per cub. ft.

= a density of 62·4 lbs. per cub. ft. approx.

238. *If W be the weight of a given substance in poundals, ρ its density in lbs. per cubic foot, V its volume in cubic feet, and g the acceleration due to gravity measured in foot-second units, then $W = g\rho V$.*

For, if M be the mass of the substance, we have by Art. 180,

$$W = Mg.$$

Also M = mass of V cubic feet of the substance

= $V \times$ mass of one cubic foot

= $V \cdot \rho$.

$\therefore W = g\rho V$.

A similar relation is true if W be expressed in dynes, ρ in grammes per cubic centimetre, V in cubic centimetres, and g in centimetre-second units.

239. Specific Gravity. Def. *The specific gravity of a given substance is the ratio of the weight of any volume of the substance to the weight of an equal volume of the standard substance.*

Thus the specific gravity is always a number.

[N.B. The term specific gravity is often shortened to sp. gr.]

For convenience the standard substance usually taken is pure water at a temperature of 4°C .

Since the weight of a cubic foot of mercury is found to be 13·596 times that of a cubic foot of water, the specific gravity of mercury is the number 13·596.

When we say that the specific gravity of gold is 19·25, water is the standard substance, and hence we mean that a cubic foot of gold would weigh 19·25 times as much as a cubic foot of water, *i.e.* about $19 \cdot 25 \times 62\frac{1}{2}$ lbs., *i.e.* about $1203\frac{1}{2}$ lbs. wt.

240. *Specific gravity of gases.* Since gases are very light compared with water, their sp. gr. is often referred to air at a temperature of 0°C . and with the mercury barometer (Art. 289) standing at a height of 30 inches. The mass of a cubic foot of air under these conditions is about 1.25 ozs.

241. The following table gives the approximate specific gravities of some important substances.

Solids.

Platinum 21.5	Glass (Crown) 2.5 to 2.7
Gold 19.25	„ (Flint) 3.0 to 3.5
Lead 11.3	Ivory 1.9
Silver 10.5	Oak .7 to 1.0
Copper 8.9	Cedar .6
Brass 8.4	Poplar .4
Iron 7.8	Cork .24

Liquids at 0°C .

Mercury 13.596	Milk 1.03
Sulphuric Acid 1.85	Alcohol .8
Glycerine 1.27	Ether .73

242. If W be the weight of a volume V of a body whose specific gravity is s , and w be the weight of a unit volume of the standard substance, then $W = V \cdot s \cdot w$.

For $s = \frac{\text{weight of a unit volume of the body}}{\text{weight of a unit volume of the standard substance}}$.

$\therefore$ wt. of unit volume of the body $= s \cdot w$.

$\therefore$ wt. of V units of volume of the body $= V \cdot s \cdot w$.

$\therefore W = V \cdot s \cdot w$.

Ex. If a cubic foot of water weigh $62\frac{1}{2}$ lbs., what is the weight of 4 cubic yards of copper, the sp. gr. of copper being 8.8?

Here $w = 62\frac{1}{2}$ lbs. wt., $V = 108$ cub. ft., and $s = 8.8$.

$$\begin{aligned}\therefore W &= 108 \times 8.8 \times 62\frac{1}{2} = 59400 \text{ lbs. wt.} \\ &= 26\frac{2}{3} \text{ tons wt.}\end{aligned}$$

EXAMPLES. XL

[In all examples it may be assumed that the weight of a cubic foot of water is $62\frac{1}{2}$ lbs.]

1. What is the weight of a cubic foot of iron (sp. gr. = 9)?
2. The sp. gr. of brass is 8; what is its density in ounces per cubic inch, given that the density of water is 1000 ozs. per cubic foot?
3. A gallon of water weighs 10 lbs. and the sp. gr. of mercury is 13.598. What is the weight of a gallon of mercury?
4. Find the weight of a litre (a cub. decimetre or 1000 cub. cms.) of mercury at the standard temperature when its sp. gr. is 13.6.
5. If 13 cubic inches of gold weigh as much as $96\frac{1}{4}$ cubic inches of quartz and the sp. gr. of gold be 19.25, find that of quartz.
6. The sp. gr. of gold being 19.25, how many cubic feet of gold will weigh a ton?
7. The sp. gr. of cast copper is 8.88 and that of copper wire is 8.79. What change of volume does a kilogramme of cast copper undergo in being drawn into wire?
8. If a foot length of iron pipe weigh 64.4 lbs. when the diameter of the bore is 4 inches and the thickness of the metal is $1\frac{1}{2}$ inches, what is the sp. gr. of the iron?
9. A rod, 18 inches long and of uniform cross-section, weighs 3 ozs. and its sp. gr. is 8.8. What fraction of a square inch is its area?
10. What volume of copper (sp. gr. = 8.9) weighs 139.0625 kilogrammes?
11. The mass of 9 cubic feet of metal is 4900 lbs.; find its density in grammes per cubic centimetre.
12. The mass of 45 cubic metres of wood is 36,000 kilogrammes; find its density in lbs. per cubic foot.

243. Specific gravities and densities of mixtures. *To find the specific gravity of a mixture of given volumes of different fluids whose specific gravities are given.*

Let $V_1, V_2, V_3, \dots$ be the volumes of the different fluids and $s_1, s_2, s_3, \dots$ their specific gravities, so that the weights of the different fluids are $V_1 s_1 w, V_2 s_2 w, V_3 s_3 w, \dots$ where w is the weight of a unit volume of the standard substance.

(1) When the fluids are mixed let there be no diminution of volume, so that the final volume is $V_1 + V_2 + V_3 + \dots$.

Let $\bar{s}$ be the new specific gravity, so that the sum of the weights of the fluids is

$$(V_1 + V_2 + V_3 + \dots) \bar{s} \cdot w.$$

Since the sum of the weights must be unaltered, we have

$$[V_1 + V_2 + V_3 + \dots] \bar{s} \cdot w = V_1 s_1 w + V_2 s_2 w + V_3 s_3 w + \dots$$

$$\therefore \bar{s} = \frac{V_1 s_1 + V_2 s_2 + V_3 s_3 + \dots}{V_1 + V_2 + V_3 + \dots}$$

(2) When there is a loss of volume on mixing the fluids together, as sometimes happens, let the final volume be n times the sum of the original volumes, where n is a proper fraction.

In this case we have

$$n [V_1 + V_2 + V_3 + \dots] \bar{s} w = V_1 s_1 w + V_2 s_2 w + \dots,$$

so that

$$\bar{s} = \frac{V_1 s_1 + V_2 s_2 + \dots}{n (V_1 + V_2 + \dots)}.$$

Similar formulae will hold if the densities instead of the specific gravities be given. For the original specific gravities $s_1, s_2, \dots$ we must substitute the original densities $\rho_1, \rho_2, \dots$ and for the final specific gravity $\bar{s}$ we must substitute the final density $\bar{\rho}$.

Ex. *Volumes proportional to the numbers 1, 2, and 3 of three liquids, whose sp. grs. are proportional to 1.2, 1.4, and 1.6, are mixed together; find the sp. gr. of the mixture.*

Let the volumes of the liquids be $x, 2x$, and $3x$. Their weights are therefore

$$wx \times 1.2, \quad 2wx \times 1.4, \quad \text{and} \quad 3wx \times 1.6.$$

Also, if $\bar{s}$ be the sp. gr. of the mixture, the total weight is

$$w\bar{s} (x + 2x + 3x).$$

Equating these two, we have

$$6w\bar{s} \cdot x = wx \times 8.8,$$

$$\therefore \bar{s} = \frac{1}{6} \times 8.8 = 1.46.$$

244. *To find the specific gravity of a mixture of given weights of different fluids whose specific gravities are given.*

Let $W_1, W_2, \dots$ be the weights of the given quantities of the fluids, $s_1, s_2, \dots$ their specific gravities, and w the weight of a unit volume of the standard substance.

By Art. 242, the volumes of the different fluids are

$$\frac{W_1}{s_1 w}, \quad \frac{W_2}{s_2 w}, \quad \dots$$

If no loss of volume takes place when the mixture is made,

the new volume is $\frac{W_1}{s_1 w} + \frac{W_2}{s_2 w} + \dots$

Hence, if $\bar{s}$ be the new specific gravity, the sum of the weights of the fluids is, by Art. 242,

$$\left(\frac{W_1}{s_1 w} + \frac{W_2}{s_2 w} + \dots \right) \bar{s} w, \text{ i.e. } \left(\frac{W_1}{s_1} + \frac{W_2}{s_2} + \dots \right) \bar{s}.$$

Hence, since the sum of the weights necessarily remains the same, we have

$$\left(\frac{W_1}{s_1} + \frac{W_2}{s_2} + \dots \right) \bar{s} = W_1 + W_2 + \dots$$

so that

$$\bar{s} = \frac{W_1 + W_2 + \dots}{\frac{W_1}{s_1} + \frac{W_2}{s_2} + \dots}.$$

If there be a contraction of volume so that the final volume is n times the sum of the original volumes, then, as in the last article, we have

$$\bar{s} = \frac{W_1 + W_2 + \dots}{n \left[\frac{W_1}{s_1} + \frac{W_2}{s_2} + \dots \right]}.$$

A similar formula gives the final density in terms of the weights and the original densities.

Ex. 10 lbs. wt. of a liquid, of sp. gr. 1.25, are mixed with 6 lbs. wt. of a liquid of sp. gr. 1.15. What is the sp. gr. of the mixture?

If w be the weight of a cubic foot of water the respective volumes of the two fluids are, by Art. 242,

$$\frac{10}{1.25 \times w} \text{ and } \frac{6}{1.15 \times w} \text{ cub. ft.}$$

Hence, if $\bar{s}$ be the required sp. gr., we have

$$\left(\frac{10}{1.25 \times w} + \frac{6}{1.15 \times w} \right) \bar{s} \cdot w = \text{total weight} = 10 + 6.$$

$$\therefore \bar{s} \left[8 + \frac{120}{23} \right] = 16,$$

$$\therefore \bar{s} = \frac{16 \times 23}{304} = \frac{23}{19} = 1.2105 \dots$$

EXAMPLES. XLI

1. The sp. gr. of a liquid being $\cdot 8$, in what proportion by volume must water be mixed with it to give a liquid of sp. gr. $\cdot 85$?

2. What is the volume of a mass of wood of sp. gr. $\cdot 5$ so that, when attached to 500 ozs. of iron of sp. gr. 7, the mean sp. gr. of the whole may be unity?

3. What weight of water must be added to 27 ozs. of a salt solution whose sp. gr. is 1.08 so that the sp. gr. of the mixture may be 1.05?

4. Three equal vessels, *A*, *B*, and *C*, are half full of liquids of densities ρ_1 , ρ_2 , and ρ_3 respectively. If now *B* be filled from *A* and then *C* from *B*, find the density of the liquid now contained in *C*, the liquids being supposed to mix completely.

5. When equal volumes of two substances are mixed the sp. gr. of the mixture is 4; when equal weights of the same substances are mixed the sp. gr. of the mixture is 3. Find the sp. grs. of the substances.

6. When equal volumes of alcohol (sp. gr. $= \cdot 8$) and distilled water are mixed together the volume of the mixture (after it has returned to its original temperature) is found to fall short of the sum of the volumes of its constituents by 4 per cent. Find the sp. gr. of the mixture.

7. A mixture is made of 7 cub. cms. of sulphuric acid (sp. gr. $= 1.843$) and 3 cub. cms. of distilled water and its sp. gr. when cold is found to be 1.615. What contraction has taken place?

8. *One hundred grammes of a liquid, whose specific gravity is $\cdot 85$, are mixed with 50 cubic centimetres of a liquid of specific gravity 1.15. If no change of volume takes place, find the specific gravity of the mixture.*

The weight of the mixture $= 100 + 50 \times 1.15 = 157.5$ grammes.

Its volume $= \frac{100}{.85} + 50 = \frac{2850}{17}$ cub. cms.

Hence the sp. gr. of the mixture $= 157.5 \div \frac{2850}{17} = .94$ approx.

9. Liquid, of volume one cubic foot and of sp. gr. $\cdot 95$, is mixed with 50 lbs. of a liquid of density 1.25. If no change of volume takes place, find the sp. gr. of the mixture.

CHAPTER XIX

PRESSURES AT DIFFERENT POINTS OF A HOMOGENEOUS LIQUID WHICH IS AT REST

245. A LIQUID is said to be homogeneous when, if any equal volumes, however small, be taken from different portions of the liquid, the masses of all these equal volumes are equal.

246. *The pressure of a heavy homogeneous liquid at all points in the same horizontal plane is the same.*

Consider two points of a liquid, P and Q , which are in the same horizontal plane.

Join PQ , and consider a small portion of the liquid whose shape is a very thin cylinder having PQ as its axis.

The only forces acting on this cylinder in the direction of the axis PQ are the two pressures on the plane ends of the cylinder.

[For all the other forces acting on this cylinder are perpendicular to PQ and therefore have no effect in the direction PQ .]

Hence, for equilibrium, these pressures must be equal and opposite.

Let the plane ends of this cylinder be taken very small, so that the pressures on them per unit of area may be taken to be constant and equal respectively to the pressures at P and Q .

Hence pressure at $P \times$ area of the plane end at P
= pressure at $Q \times$ area of the plane end at Q .

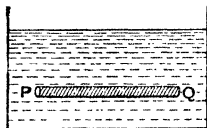
$\therefore$ pressure at P = pressure at Q .

247. *To find the pressure at any given depth of a heavy homogeneous liquid, the pressure of the atmosphere being neglected.*

Take any point P in the liquid, and draw a vertical line PA to meet the surface of the liquid in the point A .

Consider a very thin cylinder of liquid whose axis is PA . This cylinder is in equilibrium under the forces which act upon it.

The only vertical forces acting on it are its weight and



the force exerted by the rest of the liquid upon the plane face at P .

If a be the area of the plane face and x the depth AP , the weight of this small cylinder of liquid is $w \times a \times x$, where w is the weight of a unit volume of the liquid.

Also the vertical force exerted on the plane end at P is $p \times a$, where p is the pressure at P per unit of area.

Hence

$$p \cdot a = w \cdot a \cdot x.$$

$$\therefore p = w \cdot x.$$

Cor. Since the pressure at any point of a liquid depends only on the depth of the point, the necessary strength of the embankment of a reservoir depends *only* on the depth of the water, and not at all on the *area of the surface* of the water, the latter being assumed to be at rest.

248. In the above expression for the pressure care must be taken as to the units in which the quantities are measured. If British units be used, x is the depth in feet, w is the weight of a cubic foot of the liquid, and p is the pressure expressed in lbs. wt. per square foot.

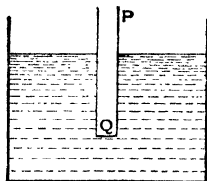
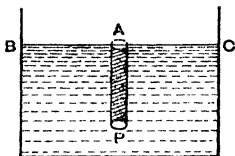
If c.g.s. units be used, x is the depth in centimetres, w is the weight of a cubic centimetre of the liquid, and p is the pressure expressed in grammes weight per square centimetre. If the liquid be water, it should be noted that now w equals the weight of one gramme. (Art. 237.)

249. The theorem of Art. 247 may be verified experimentally. PQ is a hollow cylinder one end of which, Q , is closed by a thin light flat disc which fits tightly against this end.

The cylinder and disc are then pushed into the water, the cylinder remaining always in a vertical position. The disc does not fall, being supported by the pressure of the water.

Into the upper end of the cylinder water is now poured very slowly and carefully. The disc is not found to fall until the water inside the cylinder stands at the same height as it does outside.

If h be the depth of the point Q , and A be the area of the cylinder, the pressure on Q of the external fluid must balance the weight of the internal fluid, and this weight is $A \cdot h \times w$, i.e. $A \times wh$. Hence the external pressure at Q per unit of area must be wh .



250. In Art. 247 we have neglected the pressure of the atmosphere, *i.e.* we have assumed the pressure at *A* to be zero.

If this pressure be taken into consideration and denoted by Π , the equation of that article should be

$$p \cdot a = w \cdot a \cdot x + \Pi \cdot a,$$

i.e.

$$p = wx + \Pi.$$

The pressure of the atmosphere is roughly equal to about 15 lbs. wt. per sq. inch. [This pressure is often called "15 lbs. per square inch." See Arts. 11 and 181.]

Instead of the atmospheric pressure being given in lbs. wt. per sq. inch, it is often expressed by saying that it is the same as that of a column of water, or mercury, of a given height.

This, as we shall see in Chapter XXII, is the same as telling us the height of the barometer made of that liquid. For example, if we are told that the height of the water-barometer is 34 feet, we know that the pressure of the atmosphere *per square foot* = weight of a column of water whose base is a square foot and whose height is 34 feet

$$= \text{wt. of 34 cubic feet of water}$$

$$= 34 \times 62\frac{1}{2} \text{ lbs. wt.}$$

Hence the pressure of the atmosphere per square inch

$$= \frac{34 \times 62\frac{1}{2}}{12^2} \text{ lbs. wt.}$$

$$= 14\frac{109}{144} \text{ lbs. wt.}$$

The same fact is often expressed by saying that the pressure is due to a "head" of 34 feet of water.

Ex. 1. Find the pressure in water at a depth of 222 feet, the height of the water-barometer being 34 feet.

If w be the weight of a cubic foot of water, we have

$$\Pi = w \cdot 34 \text{ lbs. wt.}$$

$$\therefore p = \Pi + wh = w \cdot 34 + w \cdot 222 = 256 w \text{ per sq. ft.}$$

$$= 256 \times 62\frac{1}{2} \text{ lbs. wt. per sq. ft.}$$

$$= \frac{16000}{144} \text{ lbs. wt. per sq. inch}$$

$$= 111\frac{1}{3} \text{ lbs. wt. per sq. inch.}$$

Ex. 2. Find the pressure at a depth of 10 metres in water, the pressure of the atmosphere being equal to that of a head of mercury (sp. gr. = 13.6) of 760 millimetres.

Here Π = pressure of the atmosphere per sq. cm. = wt. of a column of 76 cms. height = wt. of 76 cub. cms. of mercury = wt. of 76×13.6 grammes.

$\therefore p = \Pi + w \times 1000 = (76 \times 13.6 + 1000)$ grammes wt. per sq. cm., since w = wt. of 1 cub. cm. of water = wt. of 1 gramme.

$\therefore p = 2033.6$ grammes wt. per sq. cm.

251. The surface of a heavy liquid at rest is horizontal.

Take any two points, P and Q , of the liquid which are in the same horizontal plane. Draw vertical lines PA and QB to meet the surface of the liquid in A and B .

Then, by Art. 246, the pressure at P

= the pressure at Q .

Hence, by Art. 250, $\Pi + w \cdot PA = \Pi + w \cdot QB$.

$\therefore PA = QB$.

Hence, since PQ is horizontal, the line AB must be horizontal also.

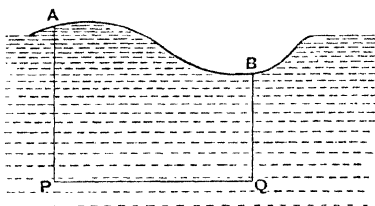
Since P and Q are any two points in the same horizontal line, it follows that any line AB drawn in the surface of the liquid must be horizontal also.

Hence the surface is a horizontal plane.

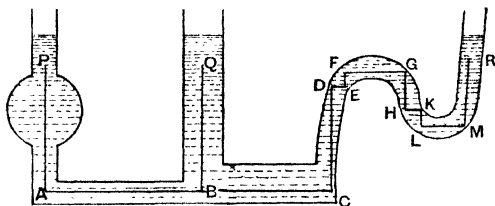
252. In the preceding proofs we have assumed that the weights of different portions of the liquids act vertically downwards in parallel directions. This assumption, as was pointed out in Art. 69, is only true when the body spoken of is small compared with the size of the earth.

If the body be comparable with the earth in size we cannot neglect the fact that, strictly speaking, the weights of the different portions of the body do not act in parallel lines but along lines directed towards the centre of the earth.

The theorem of the preceding article would not therefore apply to the surface of the sea, even if the latter were entirely at rest.



253. The proposition of Art. 246 can be proved for the case when it is impossible to connect the two points by a horizontal line which lies wholly within the liquid.



For the two points P and Q can be connected by vertical and horizontal lines such as PA , AB , and BQ in the figure.

We then have

pressure at A = pressure at B .

But pressure at A = pressure at $P + w \cdot AP$,

and pressure at B = pressure at $Q + w \cdot BQ$.

But, since P and Q are in the same horizontal plane,

$$AP = BQ.$$

Hence the pressure at P = pressure at Q .

Similarly the proposition is true for any two points at the same level. Thus, if R is at the same level as P , and vertical and horizontal lines be drawn as in the figure, we have

$$\begin{aligned} \text{pressure at } R &= \text{pressure at } P + w \cdot PA - w \cdot CD - w \cdot EF \\ &\quad + w \cdot GH + w \cdot KL - w \cdot MR \\ &= \text{pressure at } P, \end{aligned}$$

since $CD + EF + MR = AP + HG + LK.$

Hence the surface of the liquid will always stand at the same level provided the liquid be at rest.

For example, the level of the tea inside a teapot and in the spout of the teapot is always at the same height.

The statement that the surface of a liquid at rest is a horizontal plane is sometimes expressed in the form "water finds its own level."

It is this property of a liquid which enables water to be supplied to a town. A reservoir is constructed on some elevation which is higher than any part of the district to be

served. Main pipes starting from the reservoir are laid along the chief roads, and smaller pipes branch off from these mains to the houses to be supplied. If the whole of the water in the reservoir and pipes be at rest the surface of the water would, if it were possible, be at the same level in the pipes as it is in the reservoir. The mains and side-pipes may rise and fall, in whatever manner is convenient, provided that no portion of such main or pipe is higher than the surface of the water in the reservoir.

EXAMPLES. XLII

1. If a cubic foot of water weigh 1000 ozs., what is the pressure per sq. inch at the depth of a mile below the surface of the water?

2. Find the depth in water at which the pressure is 100 lbs. wt. per sq. inch, assuming the atmospheric pressure to be 15 lbs. wt. per sq. inch.

3. The sp. gr. of a certain liquid is 1.56 and the pressure at a point in the liquid is 12,090 ozs.; find the depth of the point, a foot being the unit of length.

4. The pressure in the water-pipe at the basement of a building is 34 lbs. wt. to the sq. inch, and at the third floor it is 18 lbs. wt. to the sq. inch. Find the height of this floor above the basement.

5. If the atmospheric pressure be 14 lbs. wt. per sq. inch and the sp. gr. of air be .00125, find the height of a column of air of the same uniform density which will produce the same pressure as the actual atmosphere produces.

6. If the force exerted by the atmosphere on a plane area be equal to that of a column of water 34 feet high, find the force exerted by the atmosphere on a window-pane 16 inches high and one foot wide.

7. The pressure at the bottom of a well is four times that at a depth of 2 feet; what is the depth of the well if the pressure of the atmosphere be equivalent to that of 30 feet of water?

8. If the height of the water-barometer be 34 feet, find the depth of a point below the surface of water such that the pressure at it may be twice the pressure at a point 10 feet below the surface.

9. If the pressure at a point 5 feet below the surface of a lake be one-half of the pressure at a point 44 feet below the surface, what must be the atmospheric pressure in lbs. wt. per sq. inch?

10. A vessel, whose bottom is horizontal, contains mercury whose depth is 30 inches, and water floats on the mercury to a depth of 24 inches; find the pressure at a point on the bottom of the vessel in lbs. wt. per sq. inch, the sp. gr. of mercury being 13.6.

11. A vessel is partly filled with water and then oil is poured in till it forms a layer 6 inches deep; find the pressure per sq. inch due to the weight of the liquids at a point 8·5 inches below the upper surface of the oil, assuming the sp. gr. of the oil to be ·92 and the weight of a cubic inch of water to be 252 grains.

12. A vessel contains water and mercury, the depth of the water being two feet. It being given that the sp. gr. of mercury is 13·568 and that the mass of a cubic foot of water is 1000 ozs., find, in lbs. wt. per sq. in., the pressure at a depth of two inches below the common surface of the water and mercury.

13. The lower ends of two vertical tubes, whose cross-sections are 1 and ·1 sq. inches respectively, are connected by a tube. The tube contains mercury of sp. gr. 13·596. How much water must be poured into the larger tube to raise the level in the smaller tube by one inch?

254. Resultant thrust on a plane surface. *If a plane surface be immersed in a liquid, the resultant thrust on it is equal to $wS \cdot \bar{z}$, where S is the area of the plane surface and $\bar{z}$ is the depth of its centre of gravity below the surface of the liquid, the pressure of the air being neglected.*

For consider any plane area, horizontal or inclined to the horizon, which is immersed in the liquid.

Take any small element a_1 of the plane surface situated at P , and draw PA vertically to meet the surface of the liquid in A , and let PA be z_1 .



The pressure on this small area is therefore wa_1z_1 .

(Art. 247.)

Similarly if $a_2, a_3, \dots$ be any other elements of the plane surface whose depths are $z_2, z_3, \dots$ the pressures on them are

$$wa_2z_2, \quad wa_3z_3, \quad \dots$$

Hence the resultant thrust

$$= w[a_1z_1 + a_2z_2 + \dots]. \quad (\text{Art. 49.})$$

But, if $\bar{z}$ be the depth of the centre of gravity of the given plane area, we have, as in Art. 80,

$$\bar{z} = \frac{a_1z_1 + a_2z_2 + \dots}{a_1 + a_2 + \dots},$$

$$\therefore a_1z_1 + a_2z_2 + \dots = \bar{z}(a_1 + a_2 + \dots) = \bar{z} \cdot S.$$

Hence the resultant thrust $= w\bar{z} \cdot S =$ area of the surface multiplied by the pressure at its centre of gravity, *i.e.* the resultant thrust is equal to the weight of a column of liquid whose base is equal to the area of the given plane surface, and whose height is equal to the depth below the surface of the liquid of the centre of gravity of the given plane surface.

Ex. 1. A square plate, whose edge is 8 inches, is immersed in sea-water, its upper edge being horizontal and at a depth of 12 inches below the surface of the water. Find the thrust of the water on the surface of the plate when it is inclined at 45° to the horizon, the mass of a cubic foot of sea-water being 64 lbs.

The depth of the centre of gravity of the plate

$$= 12 + 4 \cos 45^\circ = (12 + 2\sqrt{2}) \text{ inches} = \frac{6 + \sqrt{2}}{6} \text{ ft.}$$

Also the area of the plate $= \left(\frac{2}{3}\right)^2$ sq. ft.

Hence the thrust

$$= \frac{4}{9} \times \frac{6 + \sqrt{2}}{6} \times 64 \text{ lbs. wt.} = 35.149 \text{ lbs. wt. nearly.}$$

Ex. 2. A vessel in the form of a cube, each of whose edges is 2 feet, is half filled with mercury and half with water. If the sp. gr. of the mercury be 13.6, find the thrust on one of its vertical faces.

The formula of Art. 254 does not exactly suit this question, but by means of an artifice it may be made applicable.

For the thrust on the vertical face may be considered due to the thrust of *two* liquids, one, viz., water, of sp. gr. 1, filling the whole vessel, and the other of density 12.6 [*i.e.* 13.6 - 1] filling the lower half of the vessel. The required thrust is the sum of the thrusts due to the two liquids.

The thrust due to the first

$$= \text{wt. of } 2^2 \times 1 \times 1 \text{ cubic ft. of water}$$

$$= \text{wt. of 4 cubic ft. of water.}$$

The thrust due to the second.

$$= \text{wt. of } [2 \times 1] \times \frac{1}{2} \times 12.6 \text{ cubic ft. of water.}$$

$$= \text{wt. of 12.6 cubic ft. of water.}$$

The sum of these

$$= \text{wt. of 16.6 cubic ft. of water} = 16,600 \text{ ozs.} = 1037\frac{1}{2} \text{ lbs. wt.}$$

Ex. 3. A hollow cone stands with its base on a horizontal table. The area of the base of the cone is 100 sq. inches, the height is 8.64 inches, and it is filled with water. Find the thrust on the base of the cone and its ratio to the weight of the water in the cone.

The thrust = wt. of 100×8.64 cubic inches of water

$$= \frac{864}{1728} \times 1000 \text{ ozs. wt.} = 500 \text{ ozs. wt.} = 31.25 \text{ lbs. wt.}$$

Since the volume of the cone is one-third the product of the height and the area of the base, the weight of the contained water

$$= \text{wt. of } \frac{1}{3} \times 100 \times 8.64 \text{ cubic inches}$$

$$= \frac{1}{3} \times 31.25 \text{ lbs. wt.}$$

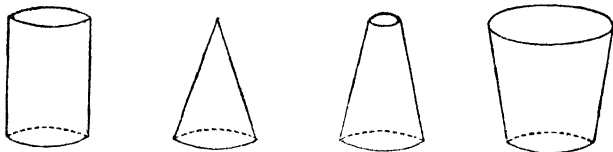
Hence, the thrust on the base of the cone

= three times the weight of the contained water.

This result, which at first sight seems impossible, is explained by the fact that the upward thrust of the base has to balance both the weight of the liquid and also the vertical component of the thrust of the curved surface of the cone upon the contained water, and this component acts downwards and could be proved to be equal to twice the weight of the contained water.

255. It follows from Arts. 246 and 247 that the pressure at any point of the base of a vessel containing liquid does not depend on the shape of the vessel, but only on the depth of the liquid.

Thus suppose we have four vessels as in the figure, of different shapes, but of the same height, and let them all be filled with water.



By Art. 246, the pressure at any point of the base is the same as at its centre, and the pressure here is, by Art. 247, just that due to the vertical height of the surface of the liquid above it.

If, in addition, the bases of the four vessels are of the same area then, by the previous article, it follows that the resultant thrusts on them are all equal.

EXAMPLES. XLIII

1. A cube, each of whose edges is 2 ft. long, stands on one of its faces on the bottom of a vessel containing water 4 ft. deep. Find the thrust of the water on one of its upright faces.

2. Water is supplied from a reservoir which is 400 ft. above the level of the sea. A tap in one of the houses supplied is at a height of 150 ft. above the sea-level and has an area of $1\frac{1}{2}$ sq. ins. Find the thrust on the tap.

3. A cube of one foot edge is suspended in water with its upper face horizontal and at a depth of $2\frac{1}{2}$ ft. below the surface. Find the thrust on each face of the cube.

4. A hole, 6 ins. sq., is made in a ship's bottom 20 ft. below the water-line. What force must be exerted to keep the water out by holding a piece of wood against the hole, assuming that a cubic ft. of sea-water weighs 64 lbs.?

5. The pressure of the atmosphere being $14\frac{1}{2}$ lbs. per sq. in., find in cwts. the thrust on a horizontal area of 7 sq. ft. placed in water at a depth of 32 ft.

6. A cubical vessel, whose side is one foot, is filled with water. Find the thrust on its surface.

7. The area of one face of a deep-sea thermometer is 54 sq. ins. Prove that when it is sunk to a depth of 6000 yards, the thrust on it is about 192 tons wt., assuming a cubic fathom of sea-water to weigh 6.15 tons.

8. Find the resultant thrust on either side of a vertical wall, whose breadth is 8 ft. and depth 12 ft., which is built in the water with its upper edge in the surface, the height of the water-barometer being 33 ft.

9. A vessel, whose base is 6 ins. sq. and whose height is 6 ins., has a neck of section 4 sq. ins. and of height 3 ins.; if it be filled with water, find the thrust on the base of the vessel.

10. The dam of a reservoir is 200 yards long and its face towards the water is rectangular and inclined at 30° to the horizon. Find the thrust acting on the dam when the water is 30 ft. deep.

Has the size of the surface of the water in the reservoir any effect on this thrust?

11. A vessel shaped like a portion of a cone is filled with water. It is one inch in diameter at the top and eight inches in diameter at the bottom and is 12 ins. high. Find the pressure in lbs. wt. per sq. in. at the centre of the base, and also the thrust on the base.

12. A square is placed in liquid with one side in the surface. Shew how to draw a horizontal line in the square dividing it into two portions, the thrusts on which are the same.

13. A vessel, in the shape of a cube whose side is one decimetre, is filled to one-third of its height with mercury whose sp. gr. is 13.6, while the rest is filled with water. Find the thrust against one of its sides in kilogrammes wt.

14. A rectangular vessel, one face of which is of height two feet and width one foot, is half filled with mercury and half with water. Find the thrust on this face, given that the sp. gr. of mercury is 13.5.

15. A cylindrical vessel, whose height is 5 ft. and diameter 1 ft., is filled with water and held so that the line joining its centre to a

point on the rim of one of its plane faces is vertical; find the thrust on each of its plane faces.

16. If in the last example the vessel is held with (1) its axis horizontal, (2) its axis vertical, find the corresponding thrusts.

17. Two equal small areas, A and B , are marked on the side of a reservoir at different depths below the surface of the water. The thrust on A is four times that on B and, if water be drawn off so that the surface of the water in the reservoir falls one foot, the thrust on A becomes nine times that on B . What were the original depths of A and B below the surface of the water?

18. A cubical box, whose edge measures 1 foot, has a pipe communicating with it which rises to a vertical height of 20 feet above the lid. It is filled with water to the top of the pipe. Find the upward thrust on the lid and the downward thrust on the base, and shew that their difference is equal to the weight of the water in the box.

How do you explain the fact that the thrust on the base is greater than the weight of the liquid it contains?

19. An artificial lake, $\frac{1}{4}$ mile long and 100 yards broad, with a gradually shelving bottom whose depth varies from nothing at one end to 88 ft. at the other, is dammed at the deep end by a masonry wall across its entire breadth. If the weight of the water be $\frac{3}{4}$ ton weight per cubic yard, prove that the thrust on the embankment is 32,266 $\frac{2}{3}$ tons weight, and that the total weight of the water in the lake is 484,000 tons.

256. *Centre of pressure of a plane area.*

If a plane be immersed in liquid the resultant fluid thrust on it is given by Art. 254.

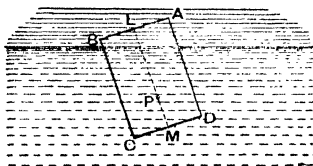
The point of the area at which it acts is called the **centre of pressure** of the given area.

The determination of the centre of pressure in any given case is a question of some difficulty.

We shall not discuss it here but shall state the position of the centre of pressure in one or two simple cases.

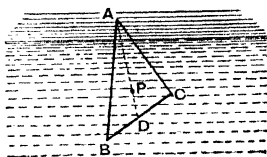
(1) A rectangle $ABCD$ is immersed with one side AB in the surface. If L and M be the middle points of AB and CD , the centre of pressure P lies on LM , and

$$LP = \frac{2}{3} LM.$$

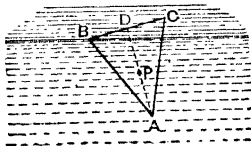


(2) A triangle ABC is immersed with an angular point A in the surface and the base BC horizontal. If D be the middle point of BC , the centre of pressure P lies on AD , and

$$AP = \frac{3}{4} AD.$$



(3) A triangle ABC is immersed with its base BC in the surface. If D be the middle point of the base, the centre of pressure P bisects DA .



Ex. 1. A rectangular hole $ABCD$, whose lower side CD is horizontal, is made in the side of a reservoir, and is closed by a door whose plane is vertical, and the door can turn freely about a hinge coinciding with AB . What force must be applied to the middle point of CD to keep the door shut if AB be one foot and AD 12 feet long, and if the water rise to the level of AB ?

If P be the required force then its moment about AB and the moment of the resultant thrust of the water about AB must be equal.

The resultant thrust of the water, by Art. 254,

$$= 1 \times 12 \times 6 \times \frac{1000}{16} \text{ lbs. wt.} = 4500 \text{ lbs. wt.}$$

Also, by (1), it acts at a point whose distance from AB

$$= \frac{2}{3} AD = 8 \text{ feet.}$$

Hence, by taking moments about AB ,

$$P \times 12 = 4500 \times 8.$$

$$\therefore P = 3000 \text{ lbs. wt.}$$

Ex. 2. A cubical box is filled with water, and is fitted with a lid $ABCD$ consisting of a uniform square plate, whose weight is two-thirds that of the contained water. If the box is held so that the lid is inclined at an angle of 45° to the horizontal, and the hinge AB is horizontal and above CD , prove that the lid is on the point of opening.

Ex. 3. A horizontal tube containing water is closed by a square lid, inclined at an angle of 30° with the horizontal and with two of its edges horizontal. If the lid be one foot square, and be movable about its lower edge as axis, and if it be on the point of opening when the water stands level with its upper edge, shew that its weight is 12.03 lbs. approximately.

CHAPTER XX

RESULTANT VERTICAL THRUST

257. If a portion of a curved surface be immersed in a heavy liquid, as in the figure of the next article, the determination of the total effect of the pressure of the liquid on it, *i.e.* of the resultant liquid thrust, is a matter of some difficulty. For the pressures at different points of the surface act in different directions and in different planes.

In the present book we shall be only concerned with the total vertical force exerted by the liquid on the curved surface. This force is called the **Resultant Vertical Thrust**. It is equal to the sum of the vertical components of the pressures which act at the different points of the given surface. For these vertical components compound, since they are parallel forces, into one single vertical force.

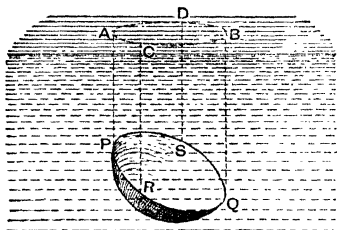
In the next article it will be shewn how this resultant vertical thrust may be found.

258. *Resultant vertical thrust on a surface immersed in a heavy liquid.*

Consider a portion of surface $PRQS$ immersed in the liquid. Through each point of the bounding edge of this surface conceive a vertical line to be drawn, and let the points in which these vertical lines meet the surface of the liquid form the curve $ACBD$.

Consider the equilibrium

of the portion of the liquid enclosed by these vertical lines, by the surface $PRQS$, and by the plane surface $ACBD$. Since, as in Art. 247, the vertical thrust of *each* small element of surface of $PSQR$ balances the weight of the corresponding thin superincumbent



cylinder of liquid, therefore the resultant of all these elementary vertical thrusts (*i.e.* the resultant vertical thrust) must be equal and opposite to, and in the same line of action as, the resultant of the weights of these thin cylinders.

But this latter resultant is the weight of the liquid $PRQSDACB$ and acts at its centre of gravity.

Also the thrust of the surface upon the liquid is equal and opposite to that of the liquid upon the surface.

Hence, *The resultant vertical thrust on any surface immersed in any heavy liquid is equal to the weight of the superincumbent liquid, and acts through the centre of gravity of this superincumbent liquid.*

259. If the liquid, instead of pressing the surface downwards, press it upwards as in the adjoining figure, the same construction should be made as in the last article.

The pressure at any point of the surface $PRQS$ depends only on the depth of the point below the surface of the liquid.

Hence, in our case, the pressure is, at any point, equal in magnitude but opposite in direction to what the pressure will be if the liquid inside the vessel be removed and instead liquid be placed outside the vessel so that AB is its surface.

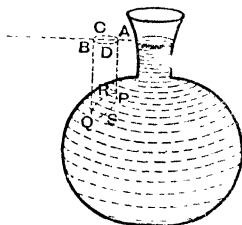
In the latter case the resultant vertical thrust will be the weight of the liquid $PQBA$.

Hence, in our case, *The resultant vertical thrust on the given portion of surface is equal to the weight of the liquid that could lie upon it up to the level of the surface of the liquid, and acts vertically upwards through the centre of gravity of this liquid.*

260. *The resultant vertical thrust on a body immersed, wholly or partly, in a liquid is equal to the weight of the liquid displaced.*

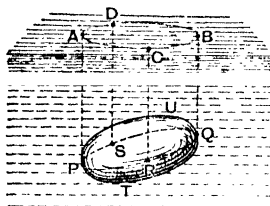
Consider the body $PTQU$ wholly immersed in a liquid.

Let a vertical line be conceived to travel round the surface of the body, touching it in the curve $PRQS$ and meeting the surface of the liquid in the curve $ACBD$.



The resultant vertical thrust on the surface $PTQRP$ is equal to the weight of the liquid that would occupy the space $PTQBA$ and acts vertically upwards through its centre of gravity.

The resultant vertical thrust on the surface $PUQRP$ is equal to the weight of the liquid that would occupy the space $PUQBA$ and acts vertically downwards through its centre of gravity.



The resultant vertical thrust on the whole body is equal to the resultant of these two thrusts, and is therefore equal to the weight of the liquid that would occupy the space $PTQU$ and acts upwards through the centre of gravity of the space $PTQU$.

Hence, *The resultant vertical thrust on a body totally immersed is equal to the weight of the displaced liquid, and acts vertically through the centre of gravity of the displaced liquid.*

This centre of gravity of the displaced liquid is often called the **centre of buoyancy** of the body.

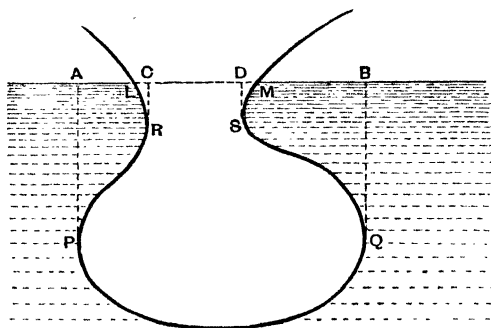
The important result just enunciated is known as the **Principle of Archimedes**, who was a Greek philosopher and lived about the year 250 B.C.

Aliter. This result may also be proved in the following manner: Conceive the body to be removed and the space $PUQRP$ to be filled up with extra liquid, the rest of the liquid being undisturbed. The pressure on each element of the surface of this extra liquid is the same as on the corresponding element of the surface of the solid (Art. 247), and hence the resultant thrust on the solid is the same as the resultant thrust on this extra liquid. But this latter balances the weight of this extra liquid. Hence etc. as before.

261. The same theorem holds if the body be partially immersed, as may be easily seen.

If the shape of the body be somewhat irregular, as in the figure on the next page, the total vertical thrust is equal to

the weight of the liquid $APQB$ acting upwards,
 less the weights of the liquids $SDBQ$ and $RCAP$ acting
 downwards,
 plus the weights of the liquids DSM and CLR acting up-
 wards,
i.e., is equivalent to the weight of the liquid that could be
 contained in $LRPQSM$ acting upwards through its centre
 of gravity. Otherwise, this is clear if, as in the previous
 article, we imagine the body removed, and the space filled
 up with extra liquid.



Floating Bodies

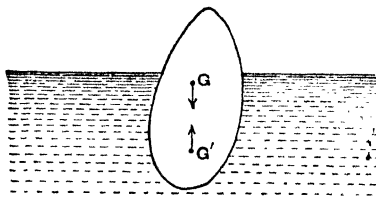
262. Conditions of equilibrium of a body freely floating in a liquid.

Consider the equilibrium of a body floating wholly or partly immersed in a liquid.

There are two, and only two, vertical forces acting on the body,

(1) its weight acting through the centre of gravity G of the body, and

(2) the resultant vertical thrust on the body which is



equal to the weight of the displaced liquid and acts through the centre of buoyancy,

i.e. the centre of gravity G' of the displaced liquid.

For equilibrium these two forces must be equal and act in opposite directions in the same vertical line.

Hence the required conditions are :

(1) The weight of the displaced liquid must be equal to the weight of the body, and

(2) The centres of gravity of the body and the displaced liquid must be in the same vertical line.

263. Ex. 1. *A cylinder of wood, of height 6 feet and weight 50 lbs., floats in water. If its sp. gr. be $\frac{3}{4}$, find how much it will be depressed if a weight of 10 lbs. be placed on its upper surface.*

Let A be the area of the section of the cylinder. Then $\times \times$

$$50 = A \cdot 6 \cdot \frac{3}{4} \cdot w = A \cdot 6 \cdot \frac{3}{4} \cdot 62\frac{1}{2}, \text{ so that } A = \frac{8}{3} \text{ sq. ft.}$$

Let x be the distance through which the wood is depressed when 10 lbs. are placed on it. The weight of the water which would occupy a cylinder, of section A and height x , must therefore be 10 lbs.

$$\therefore 10 = A \cdot x \cdot w = \frac{8}{3} \cdot x \cdot 62\frac{1}{2}, \text{ so that } x = \frac{9}{16} \text{ ft.}$$

Ex. 2. *A man, whose weight is equal to 160 lbs. and whose sp. gr. is 1.1, can just float in water with his head above the surface by the aid of a piece of cork which is wholly immersed. Having given that the volume of his head is one-sixteenth of his whole volume and that the sp. gr. of cork is .24, find the volume of the cork.*

Taking the wt. of a cubic ft. of water to be $62\frac{1}{2}$ lbs. we have, if V be the volume of the man,

$$160 = V \times \frac{11}{10} \times 62\frac{1}{2},$$

so that

$$V = \frac{128}{9} \text{ cub. ft.}$$

Again, since the weight of the man and the cork must be equal to the weight of the fluid displaced, we have, if V' be the volume of the cork in cubic feet,

$$160 + V' \times .24 \times 62\frac{1}{2} = \left(\frac{11}{16}V + V'\right) \cdot 1 \cdot 62\frac{1}{2}.$$

$$\therefore V' \times .76 \times 62\frac{1}{2} = 160 - \frac{15}{16} \cdot V \cdot 62\frac{1}{2} = 160 - \frac{15}{16} \cdot \frac{128}{9} \cdot \frac{125}{2}.$$

$$\therefore \frac{95}{2} V' = 160 - \frac{1500}{11} = \frac{260}{11}, \text{ so that } V' = \frac{104}{209} \text{ cub. ft.}$$

Ex. 3. *A loaded piece of wood and an elastic bladder containing air just float at the surface of the sea; what will happen if they be both plunged to a great depth in the sea and then released?*

The *resultant* upward thrust of a homogeneous liquid on a body is always the same, whatever be its depth below the surface of the liquid, provided that the volume of the body remains unaltered.

In the case of the wood, which we assume to be incompressible, the resultant thrust on it at a great depth is the same as at the surface, and therefore the body just floats.

In the case of the elastic bladder the pressure of the sea at a great depth compresses the bladder, and it therefore displaces much less liquid than at the surface of the sea. The resultant vertical thrust therefore is much diminished; and, as the bladder only just floated at the surface, it will now sink.

EXAMPLES. XLIV

1. A man, of weight 160 lbs., floats in water with 4 cubic inches of his body above the surface. What is his volume in cubic feet?

2. A glass tumbler weighs 8 ozs.; its external radius is $1\frac{1}{2}$ ins. and its height $4\frac{1}{2}$ ins.; if it be allowed to float in water with its axis vertical, find what additional weight must be placed on it to sink it.

3. What volume of iron must be attached to a wooden beam, of length 10 ft., breadth 2 ft., and depth 5 ins., in order to sink it?

[Sp. gr. of iron = 7.2; sp. gr. of wood = .7.]

4. A certain body just floats in water. On placing it in sulphuric acid, of sp. gr. 1.85, it requires the addition of a weight of 42.5 grammes to immerse it. Find its volume.

5. A cubic foot of air weighs 1.2 ozs. A balloon, so thin that the volume of its substance may be neglected, contains 1.5 cubic ft. of coal-gas, and the envelope together with the car and appendages weighs 1 oz. The balloon just floats in the middle of a room without ascending or descending; find the sp. gr. of the gas compared with (1) air, and (2) water.

6. The mass of a litre (*i.e.*, a cubic decimetre) of air is 1.2 grammes and that of a litre of hydrogen is .089 gramme. The material of a balloon weighs 50 kilogrammes; what must be its volume so that it may just rise when filled with hydrogen?

7. What must be the internal volume of a balloon if the whole mass to be raised is 500 lbs. (occupying 5 cubic ft.), the mass of a cubic ft. of air being .081 lb. and that of a cubic ft. of the gas with which the balloon is filled being .0054 lb.?

8. A body floats with one-tenth of its volume above the surface of pure water. What fraction of its volume would project above the surface if it floated in a liquid of sp. gr. 1.25?

9. A piece of iron weighing 275 grammes floats in mercury of sp. gr. 15.59 with $\frac{2}{3}$ ths of its volume immersed. Find the volume and sp. gr. of the iron.

10. If an iceberg be in the form of a cube and float with a height of 30 ft. above the surface of the water, what depth will it have below the surface of the water, given that the densities of ice and sea-water are as .918 to 1.025?

11. A ship, of mass 1000 tons, goes from fresh water to salt water. If the area of the section of the ship at the water line be 15,000 sq. ft. and her sides vertical where they cut the water, find how much the ship will rise, taking the sp. gr. of sea-water to be 1.026.

12. A cubical block of wood of sp. gr. whose edge is one foot, floats with two faces horizontal down a fresh water river and out to sea where a fall of snow occurs causing the block to sink to the same depth as in the river. If the sp. gr. of sea-water be 1.025, shew that the weight of the snow on the block is 20 ozs.

13. A wine-bottle, which below the neck is perfectly cylindrical and has a flat bottom, is placed in pure water and is found to float upright with $4\frac{1}{2}$ inches immersed. The bottle is now removed from the water, is dried, and immersed in oil of sp. gr. .915. How much of it will be immersed?

14. A piece of pomegranate wood, whose sp. gr. is 1.35, is fastened to a block of lignum vitæ, whose sp. gr. is .65, and the combination will then just float in water; shew that the volumes of the portions of wood are equal. *Proceed without fear*

15. A piece of cork, whose weight is 19 ozs., is attached to a bar of silver weighing 63 ozs. and the two together just float in water; if the sp. gr. of silver be 10.5, find the sp. gr. of cork. 7

16. A piece of box-wood whose sp. gr. is 1.32 is fastened to a piece of walnut-wood of sp. gr. .68 and the two together just float in water; compare the volumes of the two woods. 6

17. A rod of uniform section is formed partly of platinum (sp. gr. = 21) and partly of iron (sp. gr. = 7.5). The platinum portion being 2 ins. long, what will be the length of the iron portion when the whole floats in mercury (sp. gr. = 13.5) with one inch above the surface? 7

18. A piece of gold, of sp. gr. 19.25, weighs 96.25 grammes, and when immersed in water displaces 6 grammes. Examine whether the gold be hollow and, if it be, find the size of the cavity. 4

19. A man, whose weight is ten stone and whose sp. gr. is 1.1, just floats in water by holding under the water a quantity of cork. If the sp. gr. of the cork be .24, find its volume. 3

20. A cube of wood floating in water supports a weight of 480 ozs. On the weight being removed it rises one inch. Find the size of the cube. 1

21. A block of wood floats in liquid with $\frac{4}{5}$ ths of its volume immersed. In another liquid it floats with $\frac{2}{3}$ rds of its volume immersed. If the liquids be mixed together in equal quantities by weight, what fraction of the volume of the wood would now be immersed?

22. A piece of iron, the mass of which is 26 lbs., is placed on the top of a cubical block of wood, floating in water, and sinks it so that the upper surface of the wood is level with the surface of the water. The iron is then removed. Find the mass of the iron that must be attached to the bottom of the wood so that the top may be as before in the surface of the water.

[Sp. gr. of iron = 7.5.]

23. A cubical box, of one foot external dimensions, is made of material, of thickness one inch, and floats in water immersed to a depth of $3\frac{3}{4}$ inches. How many cubic inches of water must be poured in so that the water outside and inside may stand at the same level?

How deep in the water will the box then be?

24. A thin uniform rod, of weight W , is loaded at one end with a weight P of insignificant volume. If the rod float in an inclined position with $\frac{1}{n}$ th of its length out of the water, prove that

$$(n-1)P = W.$$

25. An ordinary bottle containing air and water floats in water neck downwards. Shew that if it be immersed in water to a sufficient depth and left to itself it will sink to the bottom. What condition determines the point at which it would neither rise nor sink?

264. *A body floats with part of its volume immersed in one liquid and with the rest in another liquid; to determine the conditions of equilibrium.*

The weight of the body must clearly be equal to the resultant vertical thrust of the two liquids, i.e. to the sum of the weights of the displaced portions of the two liquids. Also the centres of gravity of the body and of the displaced liquid must be in the same vertical line.

This includes the case of a body floating partly immersed in liquid and partly in air.

265. Ex. 1. *A vessel contains water and mercury. A cube of iron, 5 cms. along each edge, is in equilibrium in the liquids with its faces vertical and horizontal. Find how much of it is in each liquid, the specific gravities of iron and mercury being 7.7 and 13.6.*

Let x cms. be the height of the part in the mercury, and therefore $(5-x)$ cms. that of the part in the water.

Since the weight of the iron is equal to the sum of the weights of the displaced mercury and water, therefore

$$5 \times 7.7 = x \times 13.6 + (5-x) \times 1.$$

$$\therefore x = \frac{2.83}{1.9} \text{ cms.}$$

Ex. 2. A piece of wood floats in a beaker of water with $\frac{9}{10}$ ths of its volume immersed. When the beaker is put under the receiver of an air-pump and the air withdrawn, how is the immersion of the wood affected, the sp. gr. of air being .0013?

Let V be the volume of the wood, and xV the volume immersed when the air is withdrawn.

The wt. of $\frac{9V}{10}$ of water together with that of $\frac{V}{10}$ of air must equal the wt. of xV of water. For each is equal to the wt. of the wood.

$$\therefore \frac{9V}{10} \cdot 1 + \frac{V}{10} \times .0013 = xV \cdot 1.$$

$$\therefore x = .90013,$$

so that the volume immersed in water is increased from $.9V$ to $.90013V$.

EXAMPLES. XLV

1. A circular cylinder floats in water with its axis vertical, half its axis being immersed; find the sp. gr. of the cylinder if the sp. gr. of the air be .0013.

2. An inch cube of a substance, of sp. gr. 1.2, is immersed in a vessel containing two liquids which do not mix. The sp. grs. of the liquids are 1.0 and 1.5. Find how much of the solid will be immersed in the lower liquid.

3. A cub. ft. of water weighs 1000 ozs. and a cub. ft. of oil 840 ozs. The oil is poured on the top of the water without mixing and a sphere, whose volume is 36 cub. ins. and whose mass is $19\frac{1}{4}$ ozs., is placed in the mixture. How many cub. ins. of its volume will be below the surface of the water, the layer of oil being sufficiently deep for complete immersion of the sphere?

4. A uniform cylinder floats in mercury with 5.1435 ins. of the axis immersed. Water is then poured on the mercury to a depth of one inch and it is found that 5.07 ins. of the axis is below the surface of the mercury. Find the sp. gr. of the mercury.

5. A cylinder of wood floats in water with its axis vertical and having three-fourths of its length immersed. Oil, whose sp. gr. is half that of water, is then poured into the vessel to a sufficient depth to cover the cylinder. How much of the cylinder will now be immersed in the water?

6. If a body be floating partially immersed in a liquid and the air in contact with it be suddenly removed, will the body rise or sink?

7. If a body be floating partially immersed in liquid under the exhausted receiver of an air-pump and the air be suddenly admitted, will the body rise or sink?

8. A piece of wood floats partly immersed in water, and oil is poured ~~up~~ the water until the wood is completely covered. What change, if any, will this make in the volume of the portion of the wood below the water?

9. A body floats in water contained in a vessel placed under an exhausted receiver with half its volume immersed. Air is then forced into the receiver until its density is 80 times that of air at atmospheric pressure. Prove that the volume immersed in water will then be $\frac{4}{5}$ ths of the whole volume, assuming the sp. gr. of air at atmospheric pressure to be .00125.

10. A cube floats in distilled water with $\frac{4}{5}$ ths of its volume immersed. It is now placed inside a condenser where the pressure is that of ten atmospheres; find the alteration in the depth of immersion, the sp. gr. of the air at atmospheric pressure being .0013.

266. *A body rests totally immersed in a given liquid, being supported by a string; to find the tension of the string.*

The vertical upward forces acting on the body are the tension of the string, and the resultant vertical thrust of the liquid which, by Art. 260, is equal to the weight of the displaced liquid. The vertical downward force is the weight of the body.

Hence, for equilibrium, we have

Tension of the string + wt. of displaced liquid = wt. of the body, so that

Tension of the string = wt. of the body - wt. of the displaced liquid.

267. The tension of the string in the previous article is the apparent weight of the body in the given liquid, so that the apparent weight of the body in the given liquid is less than its real weight by the weight of the liquid which it displaces.

If a body of weight W and sp. gr. s be immersed in water, the weight of the water displaced is $\frac{W}{s}$, so that $\frac{W}{s}$ is the apparent loss of weight. If the body be immersed in a liquid of sp. gr. s' , the apparent loss of weight is $W \cdot \frac{s'}{s}$.

This fact is of some importance when we are "weighing" a given body by means of a balance or otherwise. To obtain a perfectly accurate result the weighing should be performed

in vacuo. Otherwise there will be a slight discrepancy arising from the fact that the quantities of air displaced by the body and by the "weights" that we use are different. Since however the weights of the displaced air are in general very small compared with that of the body this discrepancy is not very great.

If great accuracy be desired the volumes of the body weighed and the "weights" must be determined, and the weights of the displaced air allowed for at the rate of $1\frac{1}{4}$ ozs. per cubic foot.

268. Ex. *An accurate balance is completely immersed in a vessel of water. In one scale-pan some glass (sp. gr. = 2.5) is being weighed and is balanced by a one-pound weight, whose sp. gr. is 8, which is placed in the other scale-pan. Find the real weight of the glass.*

Let the real weight of the glass be W lbs. The weight of the water which the glass displaces therefore $= \frac{1}{2.5} W = \frac{2}{5} W$.

The tension of the string holding the scale-pan in which is the glass therefore

$$= W - \frac{2}{5} W = \frac{3}{5} W.$$

Again, the weight of the water displaced by the lb. wt. $= \frac{1}{8}$ lb. wt., so that the tension of the string supporting the scale-pan in which is the "weight"

$$= 1 \text{ lb. wt.} - \frac{1}{8} \text{ lb. wt.} = \frac{7}{8} \text{ lb. wt.}$$

Since the beam of the balance is horizontal, the tensions of these two strings must be the same.

$$\therefore \frac{3}{5} W = \frac{7}{8},$$

so that

$$W = \frac{35}{24} = 1\frac{11}{24} \text{ lbs. wt.}$$

This is the real weight of the glass.

EXAMPLES. XLVI

1. A body, whose wt. is 18 lbs. and whose sp. gr. is 3, is suspended by a string. What is the tension of the string when the body is suspended (1) in water, (2) in a liquid whose sp. gr. is 2?

2. Water floats upon mercury whose sp. gr. is 13, and a mass of platinum whose sp. gr. is 21 is held suspended by a string so that $\frac{1}{24}$ ths of its volume is immersed in the mercury and the remainder of its volume in the water. Prove that the tension of the string is half the weight of the platinum.

3. A vessel contains mercury and water resting on the surface of the mercury. A mass of solid gold, wholly immersed in the fluids, is held suspended in the vessel by a fine string, the volumes immersed

in the mercury and water being as 17 : 7. Prove that the tension of the string will be half the weight of the gold, the sp. grs. of gold and mercury being 19 and 13 respectively.

4. A piece of copper and a piece of silver are fastened to the ends of a string passing over a pulley, and hang in equilibrium when entirely immersed in a liquid whose sp. gr. is 1.15. Find the ratio of the volumes of the metals, the sp. grs. of silver and copper being 10.5 and 8.9.

5. A piece of silver and a piece of gold are suspended from the two ends of a balance beam which is in equilibrium when the silver is immersed in alcohol (sp. gr. = .85) and the gold in nitric acid (sp. gr. = 1.5). The sp. grs. of silver and gold being 10.5 and 19.3 respectively, find the ratio of their masses.

6. If the sp. gr. of iron be 7.6, what will be the apparent weight of 1 cwt. of iron when weighed in water, and how many lbs. of wood of sp. gr. .6 will be required to be attached to it so that the joint body may just float?

7. A gold coin, weighing half an ounce, rests at the bottom of a glass of water; find the thrust on the bottom of the glass if the sp. gr. of the coin be 18.25.

8. A solid, of weight 1 oz., rests on the bottom of a vessel of water; if the thrust of the body on the bottom be $\frac{35}{37}$ oz. find its sp. gr.

9. A compound of gold and alloy weighs 67 grains in air and 63 in water; the sp. gr. of the gold is 19 and that of the alloy is 10; find the weight of the gold in the compound.

10. The weights in air and water of a mixture composed of copper and lead are as 10 : 9; if the sp. grs. of copper and lead be 9 and 11.5, compare the weights of the metals forming the mixture.

11. A piece of lead and a piece of wood balance one another when weighed in air; which will really weigh the most and why?

12. The mass of a body *A* is twice that of a body *B*, but their apparent weights in water are the same. Given that the sp. gr. of *A* is $\frac{5}{3}$, find that of *B*.

13. A block of wood, of volume 26 cub. ins., floats in water with two-thirds of its volume immersed; find the volume of a piece of metal, whose sp. gr. is 8 times that of the wood, which, when suspended from the lower part of the wood, would cause it to be just totally immersed. When this is the case find the upward force which would hold the combined body just half immersed.

14. A cylindrical bucket, 10 ins. in diameter and one foot high, is half filled with water. A half hundred-weight of iron is suspended by a thin wire and held so that it is completely immersed in the water without touching the bottom of the bucket. Subsequently the wire is removed and the iron is allowed to rest on the bottom of the bucket. By how much will the thrust on the bottom of the bucket in each case be by the presence of the iron?

[The mass of a cubic foot of iron is 490 lbs.]

269. If a body be totally immersed in a liquid whose specific gravity is greater than that of the body, the resultant vertical thrust on the body is greater than its weight, and the body will ascend unless prevented from doing so.

Ex. 1. A piece of wood, of weight 12 lbs. and sp. gr. $\frac{3}{4}$, is tied by a string to the bottom of a vessel of water so as to be totally immersed. What is the tension of the string?

Since

$$\frac{\text{wt. of water displaced by the wood}}{\text{wt. of the wood}} = \frac{\text{sp. gr. of water}}{\text{sp. gr. of the wood}} = \frac{1}{\frac{3}{4}} = \frac{4}{3},$$

$$\therefore \text{wt. of the displaced water} = \frac{4}{3} \times 12 \text{ lbs. wt.} = 16 \text{ lbs. wt.}$$

For equilibrium we must have

Tension of the string + wt. of the wood = wt. of the displaced water.

$$\therefore \text{tension of the string} = 16 - 12 = 4 \text{ lbs. wt.}$$

Ex. 2. The mass of a balloon and the gas which it contains is 3500 lbs. If the balloon displace 48,000 cub. ft. of air and the mass of a cub. ft. of air be 1.25 ozs., find the acceleration with which the balloon commences to ascend.

$$\begin{aligned} \text{The weight of the air displaced by the balloon} &= 48000 \times 1.25 \text{ ozs. wt.} \\ &= 3750 \text{ lbs. wt.} \end{aligned}$$

Hence the upward force on the balloon

$$\begin{aligned} &= \text{wt. of displaced air} - \text{wt. of balloon} \\ &= 250 \text{ lbs. wt.} = 250g \text{ poundals.} \end{aligned}$$

$$\therefore \text{initial acceleration of balloon} = \frac{\text{moving force}}{\text{mass moved}} = \frac{250g}{3500} = \frac{g}{14}.$$

EXAMPLES. XLVII

1. A piece of cork, weighing one ounce, is attached by a string to the bottom of a vessel filled with water so that the cork is wholly immersed. If the sp. gr. of the cork be .25, find the tension of the string.

2. A piece of cork weighs 12 ozs. and its sp. gr. is .24. What is the least weight that will immerse it in water?

3. A block of wood, whose sp. gr. is .8 and weight 6 lbs., is attached by a string, which cannot bear a strain of more than 2 lbs. wt., to the bottom of a barrel partly filled with water in which the block is wholly immersed. Fluid whose sp. gr. is 1.2 is now poured into the barrel, so as to mix with the water, until the barrel is full. Prove that the string will break if the barrel were less than two-thirds full of water.

4. A block of wood, whose sp. gr. is 1.2 and whose weight is 6 lbs., is supported by a string, which cannot bear a tension of more than 1.5 lbs. wt., in a barrel partly filled with water in which the block is wholly immersed. Fluid whose sp. gr. is .7 is now poured into the

barrel, so as to mix with the water, until it is filled. Shew that the string would break if the barrel were less than two-thirds full of water.

5. A cylinder of wood, whose weight is 15 lbs. and length 3 ft., floats in water with its axis vertical and half immersed in water. What force will be required to depress it six inches more?

6. A litre of air contains 1.29 grms. and a litre of coal-gas .52 gm. A balloon contains 4,000,000 litres of coal-gas and the mass of the envelope and its appendages is 1,500,000 grms. What additional weight will it be able to sustain in the air?

7. A balloon containing 10 cub. ft. of hydrogen is prevented from rising by a string attached to it. Find the tension of the string, a cub. ft. of air being assumed to weigh 1.25 ozs. and the sp. gr. of air being 14.6 times that of hydrogen.

8. The volume of a balloon and its appendages is 64,000 cub. ft., and its mass together with that of the gas it contains is 2 tons; with what acceleration will it commence to ascend if the mass of a cub. ft. of air be 1.24 ozs.?

9. A sunk ship is raised by attaching to its sides 60 empty water-tight drums, each of capacity 100 cub. ft. and of weight 400 lbs. If they are just sufficient to cause the ship to rise, find the weight of the ship in water. The sp. gr. of sea-water is 1.03, and a cub. ft. of distilled water weighs 62.3 lbs.

270. *Conditions of equilibrium of a body partly immersed in a liquid and supported by a string attached to some point of it.*

Let P be the point of the body at which the string is attached, and let T poundals be its tension.

Let V be the volume of the body, w its wt. per unit of volume, and G its centre of gravity.

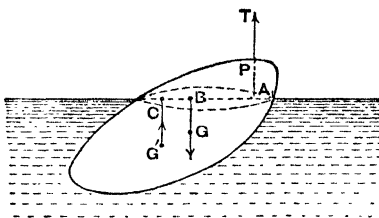
Let V' be the volume of the displaced liquid, w' its wt. per unit of volume, and G' its centre of gravity.

Let the vertical lines through P , G , and G' meet the surface of the liquid in the points A , B , and C .

The vertical forces acting on the body are

- (1) the tension T acting upwards through A ,
- (2) the weight Vw acting downwards through B ,

(Art. 238)



and (3) the resultant vertical thrust $V'w'$ acting upwards through C . (Art. 260)

Since these three forces are in equilibrium the points A , B , and C must be in the same straight line, and also, by Art. 47, we must have

$$T + V'w' = Vw \dots\dots\dots(i),$$

$$\text{and} \quad V'w' \times AC = Vw \times AB \dots\dots\dots(ii).$$

Ex. 1. A uniform rod, of length $2a$, floats partly immersed in a liquid, being supported by a string fastened to one of its ends. If the density of the liquid be $\frac{4}{3}$ times that of the rod, prove that the rod will rest with half its length out of the liquid.

Find also the tension of the string.

Let LM be the rod, N the point where it meets the liquid, G' the middle point of MN , and G the middle point of the rod.

Let w be the weight of a unit volume of the rod, and $\frac{4}{3}w$ that of the liquid.

Let the length of the immersed portion of the rod be x , and k the sectional area of the rod.

The weight of the rod is $k \cdot 2a \cdot w$, and that of the displaced liquid is $k \cdot x \cdot \frac{4}{3}w$.

If T be the tension of the string, the conditions of equilibrium are

$$T + k \cdot x \cdot \frac{4}{3}w = 2a \cdot k \cdot w \dots\dots\dots(1),$$

$$\text{and} \quad k \cdot x \cdot \frac{4}{3}w \times AC = 2a \cdot k \cdot w \times AB \dots\dots\dots(2).$$

The second equation gives

$$\frac{2x}{3a} = \frac{AB}{AC} = \frac{LG}{LG'} = \frac{a}{2a - \frac{1}{2}x},$$

$$\therefore x^2 - 4ax + 3a^2 = 0.$$

Hence $x = a$, the larger solution $3a$ of this equation being clearly inadmissible.

Hence half the rod is immersed.

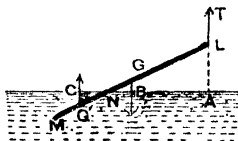
Also, substituting this value in (1), we have

$$T = \frac{2}{3}k \cdot a \cdot w = \frac{1}{3} \text{ wt. of the rod.}$$

Ex. 2. A uniform rod, six feet long, can move about a fulcrum which is above the surface of some water. In the position of equilibrium four feet of the rod are immersed; prove that its sp. gr. is $\frac{8}{9}$.

Ex. 3. A uniform rod is suspended by two vertical strings attached to its extremities and half of it is immersed in water; if its sp. gr. be 2.5, prove that the tensions of the strings will be as 9 : 7.

Ex. 4. A uniform rod capable of turning about one of its ends, which is out of the water, rests inclined to the vertical with one-third of its length in the water; prove that its sp. gr. is $\frac{8}{9}$.



Stability of equilibrium

271. When a body is floating in liquid we have shewn that its centre of gravity G and the centre of buoyancy H must be in the same vertical line. (Art. 262.)

Now let the body be slightly turned round, so that the line HG becomes inclined to the vertical. The thrust of the liquid in the new position may tend to bring the body back into its original position, in which case the equilibrium was *stable*, or it may tend to send the body still further from its original position, in which case the equilibrium was *unstable*.

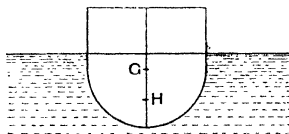


Fig. 1

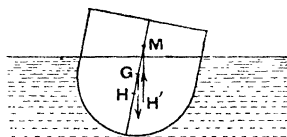


Fig. 2

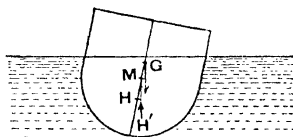


Fig. 3

The different cases are shewn in the annexed figures. Fig. 1 shews the body in its original position of equilibrium; in Figs. 2 and 3 it is shewn twisted through a small angle. In each case H' is the new centre of buoyancy, and $H'M$ is drawn vertical to meet HG in M .

In Fig. 2, where the point M is *above* G , the tendency of the forces is to turn the body in a direction opposite to that in which the hands of a watch rotate. The body will therefore return towards its original position, and the equilibrium was *stable*.

In Fig. 3, where the point M is *below* G , the tendency of the forces is opposite to that of Fig. 2. The body will therefore go further away from its original position, and the equilibrium was *unstable*.

[We have assumed that, in the above figures, the vertical line through H' meets GH ; this is generally the case for symmetrical bodies.]

It follows that the stability of the equilibrium of the above body depends on the position of M with respect to G . On account of its importance the point M has a name and is called the Metacentre. It may be formally defined as follows:

272. Metacentre. Def. *If a body float freely, and be very slightly displaced so that it displaces the same quantity of liquid as before, the point in which the vertical line through the new centre of buoyancy meets the line joining the centre of gravity of the body to the original centre of buoyancy is called the Metacentre.*

The body is in stable or unstable equilibrium according as the Metacentre is above or below the centre of gravity of the body.

It follows therefore that, to insure the stability of a floating body, its centre of gravity must be kept as low as possible. Hence we see why a ship often carries ballast, and why it is necessary to load a hydrometer (Art. 280) at its lower end.

In any given case the determination of the position of the Metacentre is a matter of considerable difficulty. This position depends chiefly on the shape of the vessel.

If the portion of the solid which is in contact with the liquid be spherical, it is clear that the pressure at each point of this spherical surface is perpendicular to the surface and so passes through the centre of the spherical surface. Hence the resultant thrust on the surface passes through the centre, which is therefore the Metacentre in this case. In this particular case the equilibrium of the body for small displacements will be stable or unstable, according as its centre of gravity is below or above the centre of the spherical portion. (Cf. Art. 90.)

Ex. A wooden ball is floating in water; shew that its equilibrium will become unstable if any weight, however small, be placed upon it at its highest point.

CHAPTER XXI

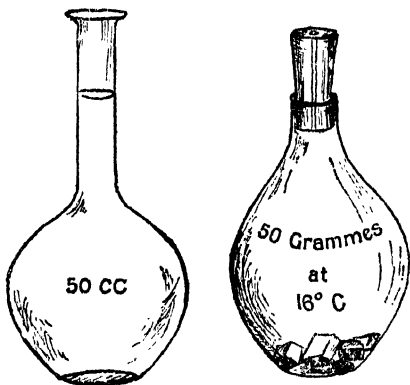
METHODS OF FINDING THE SPECIFIC GRAVITIES OF BODIES

273. In the present chapter we shall discuss some ways of obtaining the specific gravities of substances.

To find the specific gravity of any substance with respect to water we have to compare its weight with that of an equal volume of water.

The principal methods are by the use of (1) The Specific Gravity Bottle, (2) The Hydrostatic Balance, (3) Hydrometers, and (4) the U-tube. We shall consider these four in order.

274. Specific Gravity Bottle. This is a bottle capable of holding a known quantity of liquid. It is made in two forms. In (i) the neck of the bottle is open, and a mark is made on the neck up to the level of which the bottle is always exactly filled. In (ii) the bottle is closed by a tightly-fitting glass stopper, which is pierced by a small hole to allow the superfluous liquid to spirt out when the stopper is pushed home.



(1) *To find the specific gravity of a given liquid.*

Let the weight of the bottle when exhausted of air be w .

When filled with water and the stopper put in let the weight be w' .

When filled with the given liquid let its weight be w'' .

Then

$w' - w$ = weight of the water that would fill the bottle, and
 $w'' - w$ = weight of the liquid that would fill the bottle.

Since $w'' - w$ and $w' - w$ are the weights of equal quantities of the given liquid and water, therefore, by Art. 239, the sp. gr. of the liquid is

$$\frac{w'' - w}{w' - w}.$$

(2) *To find the specific gravity of a given solid which is insoluble in water.*

Let the solid be broken into pieces small enough to go into the bottle, and let the total weight of the pieces be W . Let the weight of the bottle when filled with water only be w' .

Put the solid into the bottle, fill it with water and put in the stopper, and weigh. Let the resulting weight be w'' .

Then

$W + w'$ = total weight of the solid and of the bottle when filled with water.

Also

w'' = total weight of the solid and of the bottle when filled with water

– weight of the water displaced by the solid.

Hence, by subtraction,

$W + w' - w''$ = weight of the water displaced by the solid.

Therefore W and $W + w' - w''$ are the weights of equal volumes of the solid and of water, so that the required sp. gr.

$$= \frac{W}{W + w' - w''}.$$

In performing the operations described some precautions must be taken and corrections made. The water should be at some definite temperature; a convenient temperature is 20° C.

If it were convenient the weighings should take place *in vacuo*. For, as explained in Art. 267, the air displaced by the weights and the bodies weighed has some effect on the result of a delicate experiment. In practice the weighings take place in air and corrections are applied to the results obtained.

275. If the body be, like sugar, soluble in water it must be placed in a liquid in which it is insoluble. In the case of sugar, alcohol is a suitable liquid.

Again, potassium decomposes water ; it therefore should be weighed in naphtha.

Using the notation of the last article, we have in these cases

$$\frac{\text{sp. gr. of the solid}}{\text{sp. gr. of the liquid}} = \frac{W}{W + w' - w''}.$$

276. Ex. *A sp. gr. bottle when filled with water weighs 1000 grains. If 350 grains of a powdered substance be introduced into the bottle it weighs 1250 grains. Find the sp. gr. of the powder.*

Here $1250 = 1000 + \text{wt. of substance} - \text{wt. of the water it displaces}.$

$\therefore \text{wt. of water displaced} = \text{wt. of substance} - 250 = 100 \text{ grains}.$

$\therefore \text{required sp. gr.} = \frac{\text{wt. of substance}}{\text{wt. of displaced fluid}} = \frac{350}{100} = 3.5.$

EXAMPLES. XLVIII

1. A given sp. gr. bottle weighs 7.95 grains; when full of water it weighs 187.63 grains and when full of another liquid 142.71 grains. Find the sp. gr. of the latter liquid.

2. When a sp. gr. bottle is filled with water it is counterpoised by 983 grains in addition to the counterpoise of the empty bottle, and by 773 grains when filled with alcohol; what is the sp. gr. of the latter?

3. A sp. gr. bottle, full of water, weighs 44 grms. and when some pieces of iron, weighing 10 grms. in air, are introduced into the bottle, and the bottle is again filled up with water, the combined weight is 52.7 grms. Find the sp. gr. of iron.

4. A sp. gr. bottle completely full of water weighs 38.4 grms., and when 22.3 grms. of a certain solid have been introduced it weighs 49.8 grms. Find the sp. gr. of the solid.

5. A sp. gr. bottle weighs 212 grains when it is filled with water; 50 grains of metal are put into it; the overflow of water is removed and the bottle now weighs 254 grains. Find the sp. gr. of the metal.

277. The Hydrostatic Balance. This is an ordinary balance except that it has one of its pans suspended by shorter suspending arms than the other, and that it has a hook attached to this pan to which any substance can be attached.

(1) *To find the specific gravity of a body which would sink in water.*

Let W be the weight of the body when weighed in the ordinary manner. Suspend the body by a light thread or

wire attached to the hook of the shorter arm of the scale-pan, and let the body be totally immersed in a vessel filled with water.

Put weights into the other scale-pan until the beam of the balance is again horizontal, and let w be the sum of these weights.

Then

$$\begin{aligned} w &= \text{apparent weight of the solid in water} \\ &= \text{real weight of the body} \\ &\quad - \text{the weight of the water it displaces} \\ &= W - \text{wt. of the displaced water.} \end{aligned}$$

$$\therefore W - w = \text{wt. of the displaced water.}$$

Also $W = \text{wt. of the solid.}$

$$\therefore \frac{W}{W - w} = \text{required sp. gr.}$$

If the liquid be not water, but some other liquid, then

$$\frac{W}{W - w} = \frac{\text{sp. gr. of the body}}{\text{sp. gr. of the liquid'}}$$

i.e. the ratio of the sp. grs. of the body and liquid is the ratio of the real weight of the body to the difference between the real weight of the body and its apparent weight when placed in the given liquid.

(2) *To find the specific gravity of a body which would float in water.*

In this case the body must be attached to another body, called a sinker, of such a kind that the two together would sink in water.

Let W be the weight of the body alone,

W' the weight of the sinker alone,

w the weight of the sinker and body together when placed in water,

and w' the weight of the sinker alone in water.

Then

$$\begin{aligned} w &= \text{real wt. of the sinker and body} \\ &\quad - \text{wt. of the water displaced by the sinker and body} \end{aligned}$$

(Art. 267)

$$= W + W' - \text{wt. of water displaced by the sinker and body.}$$

$$\therefore W + W' - w = \text{wt. of water displaced by the sinker and body.} \quad (1)$$

So $W' - w' = \text{wt. of water displaced by the sinker alone.}$ (2)

Hence, by subtraction,

$W - w + w' = \text{wt. of water displaced by the body alone.}$

Also $W = \text{real wt. of the body.}$

$$\therefore \frac{W}{W - w + w'} = \text{sp. gr. of the body.}$$

It will be noted that the result does not contain W' , which is the weight of the sinker, so that in practice this weight is not required.

(3) *To find the specific gravity of a given liquid.*

Take a body which is insoluble in the given liquid and in water, and let its actual weight be W .

When suspended from the short arm of the hydrostatic balance and placed in water, let its apparent weight be w .

When the given liquid is substituted for water, let the apparent weight be w' .

Then $w = \text{wt. of the body} - \text{wt. of the water it displaces.}$

Hence wt. of the water displaced $= W - w$.

So wt. of the liquid displaced $= W - w'$.

Hence $W - w'$ and $W - w$ are the weights of equal volumes of the liquid and water.

$$\therefore \frac{W - w'}{W - w} = \text{required sp. gr.}$$

278. Ex. 1. *A piece of copper weighs 9000 grms. in air and 7987.5 grms. when weighed in water. Find its specific gravity.*

Here

$7987.5 = 9000 - \text{wt. of water displaced by the copper.}$

$\therefore \text{wt. of displaced water} = 1012.5.$

$$\therefore \text{required sp. gr.} = \frac{9000}{1012.5} = 8.8.$$

Ex. 2. *A piece of cork weighs 30 grms. in air; when a piece of lead is attached the combined weight in water is 6 grms.; if the weight of the lead in water be 96 grms., what is the sp. gr. of the cork?*

If w be the wt. of lead in air, the wt. of water displaced by the lead and cork $= w + 30 - \text{combined wt. in water} = w + 30 - 6 = w + 24.$

So wt. of water displaced by the lead $= w - 96.$

Hence the wt. of water displaced by the cork

$$= (w + 24) - (w - 96) = 120.$$

$$\therefore \text{sp. gr. of the cork} = \frac{30}{120} = \frac{1}{4}.$$

Ex. 9. If a ball of platinum weigh 20·86 ozs. in air, 19·86 in water, and 19·36 in sulphuric acid, find the sp. grs. of the platinum and the sulphuric acid.

Wt. of the water displaced by the platinum
 $= 20·86 - 19·86 = 1 \text{ oz.}$

Wt. of the sulphuric acid displaced by the platinum
 $= 20·86 - 19·36 = 1·5 \text{ ozs.}$

Hence the sp. gr. of the platinum $= \frac{20·86 \text{ ozs.}}{1 \text{ oz.}} = 20·86,$

and the sp. gr. of the sulphuric acid $= \frac{1·5 \text{ ozs.}}{1·0 \text{ oz.}} = 1·5.$

EXAMPLES. XLIX

[The sp. gr. of the air is neglected.]

1. If a body weigh 732 grms. in air and 252 grms. in water, what is its sp. gr.?

2. A piece of flint-glass weighs 2·4 ozs. in air and 1·6 ozs. in water; find its sp. gr.

3. A piece of cupric sulphate weighs 3 ozs. in air and 1·86 ozs. in oil of turpentine. What is the sp. gr. of the cupric sulphate, that of oil of turpentine being ·88?

4. It is required to find the sp. gr. of potassium which decomposes water. A piece of potassium weighing 432·5 grms. in air is weighed in naphtha, the sp. gr. of which is ·847, and is found to weigh 9 grms. What is the sp. gr. of potassium?

5. A piece of lead weighs 30 grains in water. A piece of wood weighs 120 grains in air, and, when the lead is fastened to the wood, the two together weigh 20 grains in water. Find the sp. gr. of the wood.

6. A solid, which would float in water, weighs 4 lbs., and, when the solid is attached to a heavy piece of metal, the whole weighs 6 lbs. in water; the weight of the metal in water being 8 lbs., find the sp. gr. of the solid.

7. A body, of weight 300 grms. and of sp. gr. 5, has 200 grms. of another body attached to it, and the joint weight in water is 200 grms. Find the sp. gr. of the attached substance.

8. A piece of glass weighs 47 grms. in air, 22 grms. in water, and 25·8 grms. in alcohol. Find the sp. gr. of the alcohol.

9. A bullet of lead, whose sp. gr. is 11·4, weighs 1·09 ozs. in air and 1 oz. in olive oil. Find the sp. gr. of the olive oil.

10. A ball of glass weighs 665·8 grains in air, 465·8 grains in water, and 297·6 grains in sulphuric acid. What is the sp. gr. of the latter?

11. A piece of marble, of sp. gr. 2·84, weighs 92 grms. in water and 98·5 grms. in oil of turpentine. Find the sp. gr. of the oil and the volume of the marble.

12. A body is weighed in two liquids, the first of sp. gr. $\cdot 8$ and the second of sp. gr. $1\cdot 2$. In the two cases its apparent weights are 18 and 12 grms. respectively. Find its true weight and sp. gr.

13. The apparent weight of a sinker when weighed in water is 5 times the weight *in vacuo* of a portion of a certain substance; the apparent weight of the sinker and substance together is 4 times the same weight; find the sp. gr. of the substance.

14. A given body weighs 4 times as much in air as in water, and one-third as much again in water as in another liquid. Find the sp. gr. of the latter liquid.

15. The crown used by the Stuart Sovereigns which was destroyed in the 17th century was said to have been made of pure gold (sp. gr. $= 19\cdot 2$) and to have weighed $7\frac{1}{2}$ lbs. How much did it weigh in water?

If it had been of alloy, partly silver (sp. gr. $= 10\cdot 5$) and partly gold, and had weighed $7\frac{1}{4}$ lbs. in water, how much of each metal would it have contained?

279. Hydrometers. A hydrometer is an instrument which, by being floated in any liquid, determines the sp. gr. of the liquid. There are various forms of the hydrometer; we shall consider two, viz. the Common Hydrometer and Nicholson's Hydrometer.

280. Common Hydrometer. This consists of a straight glass stem ending in a bulb, or bulbs, the lower of which is loaded with mercury to make the instrument float with its stem vertical.

To find the specific gravity of a given liquid.

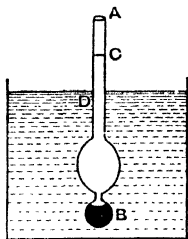
When immersed in the given liquid, let the instrument float with the point *D* of the stem at the surface of the liquid.

When immersed in water, let it float with the point *C* of the stem in the surface of the water.

Let *V* be the total volume of the instrument, and *a* the area of the section of the stem, this section being constant throughout the stem.

When immersed in the given liquid, the portion of the stem, whose length is *AD* and whose volume is *a . AD*, is out of the liquid. The volume immersed is therefore

$$V - a . AD.$$



Similarly, when placed in water, the volume immersed is

$$V - a \cdot AC.$$

In each case the weights of the displaced liquids are equal to the weight of the instrument, so that the weights of the liquids are the same.

Hence, if s be the sp. gr. of the liquid, we have

$$s(V - a \cdot AD) = V - a \cdot AC.$$

$$\therefore s = \frac{V - a \cdot AC}{V - a \cdot AD}.$$

In practice the instrument maker sends out the common hydrometer graduated by marking along its stem at different points the sp. grs. of the liquids in which the given instrument would sink to these points.

A hydrometer to shew the sp. grs. of liquids of all densities would have to be inconveniently long. Hydrometers are therefore usually made in sets to be used for liquids specifically lighter than water, for medium liquids, and for very heavy liquids respectively.

281. Ex. 1. *The whole volume of a common hydrometer is 6 cubic inches and the stem, which is a square, is $\frac{1}{8}$ inch in breadth; it floats in one liquid with 2 inches of its stem above the surface, and in another with 4 inches above the surface. Compare the specific gravities of the liquids.*

In the first liquid the volume immersed

$$= 6 - 2 \cdot \frac{1}{8^2} = \frac{191}{32} \text{ cub. ins.}$$

In the second liquid the volume immersed

$$= 6 - \frac{4}{8^2} = \frac{190}{32} \text{ cub. ins.}$$

Hence, if s_1 and s_2 be the required specific gravities, we have

$$\frac{191}{32} \cdot s_1 = \frac{190}{32} \cdot s_2.$$

$$\therefore s_1 : s_2 :: 190 : 191.$$

Ex. 2. *The stem of a common hydrometer is cylindrical, and the highest graduation corresponds to a specific gravity of 1, whilst the lowest corresponds to 1.2. What specific gravity will correspond to a point exactly midway between these divisions?*

Let V be the volume of that portion of the hydrometer which is below the highest graduation, W its weight, A the area of the section

of its stem, x the distance between its extreme graduations and w the wt. of a unit volume of water, so that

$$V \cdot 1 \cdot w = W \dots\dots\dots(1),$$

$$(V - A \cdot x) \frac{1}{6} w = W \dots\dots\dots(2).$$

$$\therefore V = \frac{W}{w}, \text{ and } A \cdot x = \frac{1}{6} \frac{W}{w}.$$

Hence, if s be the required sp. gr., we have

$$[V - \frac{1}{2} A \cdot x] s \cdot w = W.$$

$$\therefore s = \frac{W}{(V - \frac{1}{2} A x) w} = \frac{W}{W - \frac{1}{2} W} = \frac{1}{\frac{1}{2}} = 1.09.$$

This, it will be noted, is not half way between 1 and 1.2. On this hydrometer the distances between marks denoting equal increments of sp. gr. are not equal.

282. Nicholson's Hydrometer. This hydrometer consists of a hollow metal vessel A which supports by a thin stem a small pan B on which weights can be placed. At its lower end is a small heavy cup or basket C , which should be heavy enough to insure stable equilibrium when the instrument is floated in a liquid.

The instrument may be used to compare the sp. grs. of two liquids, and also to find the sp. gr. of a solid.

On the stem is a well-defined mark D , and the method consists of loading the instrument till it sinks so that this mark is in the surface of the two liquids to be compared.

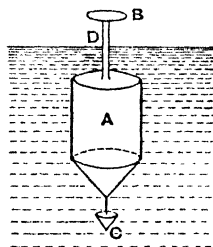
(1) *To find the sp. gr. of a given liquid.*

Let W be the weight of the instrument. Let w be the weight that must be placed on the pan B , so that the point D of the instrument may float in the level of the given liquid.

Let w_1 be the weight required when water is substituted for the given liquid.

In the first case it follows, by Art. 262, that $W + w$ is the weight of the liquid displaced by the instrument.

So $W + w_1$ is the weight of the water displaced by the instrument.



Hence $W + w$ and $W + w_1$ are the weights of equal volumes of the given liquid and water.

The required sp. gr. therefore = $\frac{W + w}{W + w_1}$.

(2) *To find the sp. gr. of a solid body.*

Let w_1 be the weight which, when placed in the pan B , will sink the instrument in water to the point D .

Place the solid upon the pan, and let the weight now required to sink the instrument to D be w_2 .

The weight of the solid therefore = $w_1 - w_2$.

Now place the solid in the cup C underneath the water, and let w_3 be the weight that must be placed in B to sink the instrument to D .

The wt. of the solid together with w_2 has therefore the same effect as the wt. of the solid in water together with w_3 .

$\therefore$ wt. of the solid + w_2

= wt. of the solid in water + w_3 .

$\therefore w_3 - w_2$ = wt. of the solid - wt. of the solid in water
= wt. of the water displaced by the solid.

(Art. 266.)

Also $w_1 - w_2$ = wt. of the solid.

$\therefore \frac{w_1 - w_2}{w_3 - w_2}$ = the required sp. gr.

It will be noted that a Nicholson's Hydrometer always displaces a constant *volume* of liquid, whilst the Common Hydrometer always displaces a constant *weight* of liquid.

263. Ex. *A Nicholson's Hydrometer when loaded with 200 grains in the upper pan sinks to the marked point in water; a stone is placed in the upper pan and the weight required to sink it to the same point is 80 grains; the stone is then placed in the lower pan and the weight required is 128 grains; find the sp. gr. of the stone.*

The weight of the hydrometer being W grains, the weight of the water displaced is equal to (i) $W + 200$, (ii) $W + 80$ + wt. of stone, and (iii) $W + 128$ + wt. of stone in water.

$\therefore W + 200 = W + 80$ + wt. of stone

= $W + 128$ + wt. of stone in water.

$$\begin{aligned}
 \therefore 120 &= \text{wt. of stone} \dots\dots\dots(1), \\
 72 &= \text{wt. of stone in water} \\
 &= 120 - \text{wt. of water displaced by stone} \dots\dots\dots(2). \\
 \therefore \text{required sp. gr.} &= \frac{\text{wt. of stone}}{\text{wt. of water displaced by stone}} = \frac{120}{120 - 72} \\
 &= \frac{120}{48} = 2.5.
 \end{aligned}$$

EXAMPLES. I

1. A common hydrometer weighs 2 ozs., and is graduated for sp. grs. varying from 1 to 1.2. What should be the volumes in cubic ins. of the portions of the instrument below the graduations 1, 1.1, and 1.2 respectively?

2. When a common hydrometer floats in water $\frac{9}{10}$ ths of its volume is immersed, and when it floats in milk $\frac{9.0}{10.3}$ of its volume is immersed; what is the sp. gr. of milk?

3. A common hydrometer has a small portion of its bulb rubbed off from frequent use. In consequence when placed in water it appears to indicate that the sp. gr. of water is 1.002; find what fraction of its weight has been lost, if s be the specific gravity of the substance of which the bulb is made.

4. A Nicholson's Hydrometer weighs 8 ozs. The addition of 2 ozs. to the upper pan causes it to sink in one liquid to the marked point, while 5 ozs. are required to produce the same result in another liquid. Compare the sp. grs. of the two liquids.

5. A Nicholson's Hydrometer, of weight $4\frac{3}{4}$ ozs., requires weights of 2 and $2\frac{3}{8}$ ozs. respectively to sink it to the fixed mark in two different liquids. Compare the sp. grs. of the two liquids.

6. A Nicholson's Hydrometer is of weight $3\frac{3}{4}$ ozs., and a weight of $1\frac{3}{4}$ ozs. is necessary to sink it to the fixed mark in water. What weight will be required to sink it to the fixed mark in a liquid of density 2.2?

7. A certain solid is placed in the upper cup of a Nicholson's Hydrometer, and it is then found that 12 grains are required to sink the instrument to the fixed mark; when the solid is placed in the lower cup 16 grains are required, and when the solid is taken away altogether 22 grains are required; find the sp. gr. of the solid.

8. The standard weight of a Nicholson's Hydrometer is 1250 grains. A small substance is placed in the upper pan, and it is found that 520 grains are needed to sink the instrument to the standard point; when the substance is placed in the lower pan, 620 grains are required. What is the sp. gr. of the substance?

9. In a Nicholson's Hydrometer if the sp. gr. of the weights be 8, what weight must be placed in the lower pan to produce the same effect as 2 ozs. in the upper pan?

284. If two liquids do not mix there is another method, by using a bent tube, of comparing their sp. grs.

ABC is a bent tube having a uniform section and straight legs.

The two liquids are poured into the two legs and rest with their common surfaces at D , and with the surfaces in contact with the air at P and Q .

Let D be in the leg BC , and E the point of the leg AB which is at the same level as D .

Let s_1 and s_2 be the sp. grs. of the fluids.

If w be the weight of a unit volume of the standard substance, the pressures at E and D are respectively

$$s_1 \cdot w \cdot EP + \Pi \quad \text{and} \quad s_2 \cdot w \cdot DQ + \Pi,$$

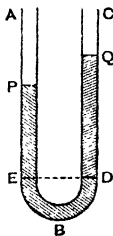
where Π is the pressure of the air.

For equilibrium these two pressures must be the same.

$$\therefore s_1 \cdot w \cdot EP + \Pi = s_2 \cdot w \cdot DQ + \Pi.$$

$$\therefore \frac{s_1}{s_2} = \frac{DQ}{EP},$$

i.e. the sp. grs. of the two fluids are inversely as the heights of their respective columns above the common surface.



EXAMPLES. LI

1. The lower portion of a U-tube contains mercury. How many inches of water must be poured into one leg of the tube to raise the mercury one inch in the other, assuming the sp. gr. of mercury to be 13.6?

2. Water is poured into a U-tube, the legs of which are 8 inches long, till they are half full. As much oil as possible is then poured into one of the legs. What length of the tube does it occupy, the sp. gr. of oil being $\frac{2}{3}$?

3. A uniform bent tube consists of two vertical branches and of a horizontal portion uniting the lower ends of the vertical portions. Enough water is poured in to occupy 6 inches of the tube and then enough oil to occupy 5 inches is poured in at the other end. If the sp. gr. of the oil be $\frac{4}{3}$, and the length of the horizontal part be 2 inches, find where the common surface of the oil and water is situated.

CHAPTER XXII

PRESSURE OF GASES

285. We have pointed out in Art. 224 that the essential difference between gases and liquids is that the latter are practically incompressible whilst the former are very easily compressible.

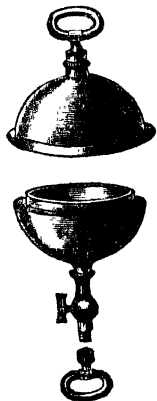
The pressure of a gas is measured in the same way as the pressure of a liquid. In the case of a liquid the pressure is due to its weight and to any external pressure that may be applied to it. In the case of a gas, however, the external pressure to which the gas is subjected is, in general, the chief cause which determines the amount of its pressure.

286. Air exerts a pressure. This may be seen from several experiments:

(1) If we put a bladder of air into the receiver of an air-pump and exhaust the air from the receiver, the pressure of the air inside the bladder is no longer counteracted by that of the air outside the bladder; the bladder increases in size and finally bursts.

(2) If we push a glass tumbler mouth downwards into water, it will be seen that the level of the water inside the tumbler is lower than that outside; the pressure of the imprisoned air forces the water down.

(3) An experiment known as that of the Magdeburg hemispheres was first made in the middle of the seventeenth century. Two hollow hemispheres were fitted together very accurately, so that they were air-tight. The hollow inside was exhausted of air, and it was then found that very great force was necessary to separate the hemispheres. If they are a foot in diameter, the force to be exerted on each is more than 1600 lbs. wt.



Air has weight. This may be shewn experimentally as follows:

Take a hollow glass globe, closed by a stopcock, and by means of an air-pump (Art. 314) or otherwise (*e.g.* by boiling a small quantity of water in the globe) exhaust it of air, and weigh the globe very carefully.

Now open the stopcock, and allow air at atmospheric pressure to enter the globe, and again weigh the globe very carefully.

The globe appears to weigh more in the second case than it does in the first case. This increase in the weight is due to the weight of the air contained in the globe.

The sp. gr. of air referred to water is found to be $\cdot 001293$, *i.e.* the weight of a cubic foot of air is about $1\cdot 293$ ounces.

It may be similarly shewn that any other gas has weight and pressure.

The following are the specific gravities of some principal gases at 0° C. and a pressure of 76 cms. of mercury.

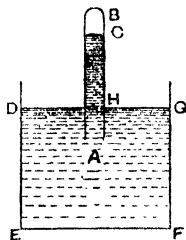
Air	$\cdot 001293$.	Oxygen	$\cdot 001430$.
Hydrogen	$\cdot 000089$.	Nitrogen	$\cdot 001256$.
Carbonic Acid	$\cdot 001977$.		

Hydrogen is hence about one-fourteenth as heavy as air, and thus is very useful for filling balloons; Carbonic acid gas is much heavier than air, and it may be poured like a liquid from one vessel into another.

287. Pressure of the atmosphere.

Take a glass tube, 3 or 4 feet long, closed at one end *B* and open at the other end *A*. Fill it carefully with mercury. Invert it and put the open end *A* into a vessel *DEFG* of mercury, whose upper surface is exposed to the atmosphere. Let the tube be vertical.

The mercury inside the tube will be found to descend till the surface of the mercury inside the tube is at a point *C* whose height above the level *H* of the mercury in the vessel is about 29 or 30 inches.



For clearness suppose the height to be 30 inches. The pressure on each square inch just inside the tube at *H* is

therefore equal to the weight of the superincumbent 30 inches of mercury.

But the pressure at *H* just inside the tube is equal to the pressure at the surface of the mercury in the vessel, which again is equal to the pressure of the atmosphere in contact with it.

Hence in our case the pressure of the atmosphere is the same as that produced by a column of mercury 30 inches high.

This experiment is often referred to as Torricelli's experiment.

If the height of the column of mercury inside the tube be carefully noted, it will be found to be continually changing. Hence it follows that the pressure of the atmosphere is continually changing. It is, in general, less when the atmosphere has in it a large quantity of vapour.

288. The pressure of the atmosphere may, when the height of the column *HC* is known, be easily expressed in lbs. wt. per sq. foot or sq. inch.

For the density of pure mercury is 13·596 times that of water, *i.e.* it is 13,596 ounces per cubic foot.

When the height of the column *HC* is 30 inches the pressure of the atmosphere per sq. inch

= wt. of 30 cubic inches of mercury

$$= 30 \times \frac{13596}{1728 \times 16} \text{ lbs. wt.}$$

$$= 14\cdot75 \dots \text{lbs. wt.}$$

Similarly in centimetre-gramme units, if the height of the column be 76 cms., the pressure of the atmosphere per sq. cm. = wt. of 76 cub. cms. of mercury

= wt. of $76 \times 13\cdot596$ cub. cms. of water

= $76 \times 13\cdot596$ grammes wt.

= 1033·296 grammes wt.

= 1013663·376 dynes.

289. Barometer. The barometer is an instrument for measuring the pressure of the air. In its simplest form it consists of a tube and reservoir similar to those used in the experiment of Art. 287, and contains liquid supported by

the pressure of the air. This pressure is measured by the height of the liquid inside the tube above the level of the liquid in the reservoir.

The liquid generally employed is mercury, on account of its great density. Glycerine is sometimes used instead.

The ordinary height of the mercury barometer is between 29 and 30 inches. If water were used, the height would be about 33 to 34 feet.

Standard atmospheric pressure. The atmosphere is said to be at standard pressure when the height of the column of the mercury barometer is 76 centimetres and the temperature is 0°C . This corresponds to a pressure, as in the previous article, of about 1,013,663 dynes per square centimetre.

In England the height for the standard pressure is usually taken to be 30 inches (≈ 76.2 cms. nearly). The corresponding pressure is 14.75 lbs. wt. per sq. inch.

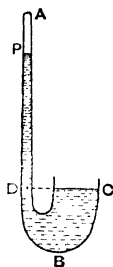
290. Siphon Barometer. The usual form of a barometer in practice is a bent tube ABC , the diameter of the long part AB being considerably smaller than that of the short part BC . It is placed so that the two portions of the tube are vertical.

The end of the short limb is exposed to the atmosphere, and the end A of the long limb is closed. The long limb is usually about 3 feet long, and inside the tube is a quantity of mercury. Above the mercury in the long limb there is a vacuum.

When the surfaces of the mercury in the two limbs are at P and C respectively, the pressure of the air is measured by the weight of a column of mercury whose height is equal to the vertical distance between P and C , *i.e.* to the vertical distance PD where D is a point on the long limb at the same level as C .

For, since there is a vacuum above P , the pressure at D is equal to the weight of a column of mercury of height DP .

Again, since C and D are at the same level the pressures at these two points are the same; also the pressure of the mercury at C is equal to the pressure of the atmosphere. Hence the pressure of the atmosphere is equal to the weight of the column DP .



The limb DP is marked at regular intervals with numbers shewing the height of the barometer corresponding to each graduation.

291. Graduation of a barometer. In graduating a barometer there is one important point to be taken into consideration, and that is that if the level of the mercury in BA rises the level of that in BC must fall. The required height of the barometric column is always the difference between these two levels.

Suppose the section of the part BA to be uniform and equal to $\frac{1}{10}$ th of a square inch, and that the section of the short limb near C is uniform and equal to one square inch.

Also suppose that the level of the mercury in the long limb appears to rise one inch. Since the increase of the volume of mercury in one limb corresponds to a decrease in the other, it follows that the level of the mercury in the short limb has fallen $\frac{1}{10}$ th of an inch.

Hence the height between the two levels has increased by $(1 + \frac{1}{10})$, or $\frac{11}{10}$ ths of an inch. Therefore an apparent increase of one inch in the height of the mercury does, in our case, correspond to a real increase of $\frac{10}{11}$ ths of an inch.

So an apparent increase of $\frac{10}{11}$ inch corresponds to a real increase of one inch.

To avoid the trouble of having to make this correction, the limb BA is divided into intervals of $\frac{10}{11}$ inch, and the markings are made as if these intervals are really inches.

More generally. Let the long limb be of sectional area A and the short limb of sectional area A' , and suppose both A and A' to be constant.

A rise of x in the level of the mercury in the long limb would cause a fall of $\frac{A}{A'}x$ in the short limb.

Hence an apparent increase of x in the height of the barometric column would correspond to a real increase of $x + \frac{A}{A'}x$, i.e. of $\frac{A + A'}{A'}x$.

So an apparent increase of $\frac{A'}{A + A'}x$ would correspond to a real increase of x .

Hence, to insure correctness, the distances between the successive graduations in the long limb are shorter than they are marked in the ratio $A' : A + A'$.

Since the sp. gr. of mercury, like other liquids, alters with its temperature, the latter must be observed in making an accurate reading of the barometer. The length of the corresponding column at the standard temperature 0°C . can then be calculated.

EXAMPLES. LII

1. At the bottom of a mine a mercurial barometer stands at 77.4 cms.; what would be the height of an oil barometer at the same place, the sp. grs. of mercury and oil being 13.596 and .9?

2. If the height of the water barometer be 1033 cms., what will be the pressure on a circular disc whose radius is 7 cms. when it is sunk to a depth of 50 metres in water?

3. When the barometer is standing at 30 ins., find the total pressure of the air on the surface of a man, assuming that the area of this surface is 18 sq. ft. and that the sp. gr. of mercury is 13.596.

4. Glycerine rises in a barometer tube to a height of 26 ft. when the mercury barometer stands at 30 ins. The sp. gr. of mercury being 13.6, find that of glycerine.

If an iron bullet be allowed to float on the mercury in a barometer, how would the height of the mercury be affected?

5. The diameter of the tube of a mercurial barometer is 1 cm. and that of the cistern is 4.5 cms. If the surface of the mercury in the tube rise through 2.5 cms., find the real alteration in the height of the barometer.

6. The diameter of the tube of a mercurial barometer is $\frac{1}{8}$ in. and that of the cistern is $1\frac{1}{2}$ ins. When the surface of the mercury rises 1 in., find the real alteration in the height of the barometer.

292. *Connection between the pressure and density of a gas.*

It is easy to shew that the density of a gas alters when its pressure alters.

Take an ordinary glass tumbler and immerse it mouth downwards in water, always taking care to keep it vertical. As the tumbler is pushed down into the water the latter rises inside the tumbler, shewing that the volume of the air has been reduced.

The pressure of the contained air, being equal to the pressure of the water with which it is in contact, is greater than the pressure at the surface of the water. Also the

pressure at the surface of the water is equal to atmospheric pressure, which was the original pressure of the contained air. Hence we see that whilst the contained air is compressed its pressure is increased.

Consider again the case of a boy's pop-gun. To expel the bullet the boy sharply pushes in the piston of the gun, thereby reducing the volume of the air considerably; since the bullet is expelled with some velocity, the pressure of the air behind it must have been increased when the volume of the air was reduced.

As another example, take a bladder with very little air in it, but tied so that this air cannot escape. Place the bladder under the receiver of an air-pump, and exhaust the air. As the air gets drawn out, its pressure on the bladder becomes less; the air inside the bladder is therefore subject to less pressure, and in consequence expands and causes the bladder to swell out.

293. The relation between the pressure and the volume of a gas is given by an experimental law known as **Boyle's Law**, which says that

The pressure of a given quantity of gas, whose temperature remains unaltered, varies inversely as its volume.

Hence, if the volume of a given quantity of gas be doubled, its pressure is halved; if the volume be trebled, its pressure is one-third of what it was originally; if its volume be halved, its pressure is doubled.

In general, if its volume be increased n times, the corresponding pressure becomes divided by n . This is the meaning of the expression "varies inversely as."

This law is generally known on the Continent under the name of Marriotte's Law.

294. In the case of air the law may be verified experimentally as follows:

ABC is a bent tube of uniform bore of which the arms BA and BC are straight. The arm BC is much longer than BA .

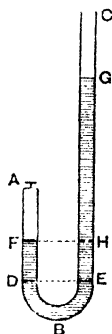
At A let there be a small plug or cap which can be screwed in so as to render the tube BA air-tight.

First let this cap be unscrewed. Pour in mercury at C until the surface is at the same level D and E in the two tubes.

Screw in the cap at *A* tightly so that a quantity of air is enclosed at atmospheric pressure.

Pour in more mercury at *C* until the level of the mercury in the longer arm stands at *G*. The level of the mercury in the shorter arm will be found to have risen to some such point as *F*, which however is below *G*. It follows that the air in the shorter arm has been diminished in volume.

Let *h* be the height of the mercury barometer at the time, and let *H* be the point on the longer tube at the same level as *F*. Then the pressure of the enclosed air



= pressure at *F*

= pressure at *H*

= wt. of column *HG* + pressure at *G*

= wt. of column *HG* + wt. of column *h*

= wt. of a column (*HG* + *h*).

$$\therefore \frac{\text{final pressure}}{\text{original pressure}} = \frac{\text{wt. of a column } (HG + h)}{\text{wt. of a column } h} = \frac{HG + h}{h}.$$

$$\text{Also } \frac{\text{original volume of the air}}{\text{final volume of the air}} = \frac{DA}{FA}.$$

It is found, when careful measurements are made, that

$$\frac{HG + h}{h} = \frac{DA}{FA}.$$

$$\therefore \frac{\text{final pressure}}{\text{original pressure}} = \frac{\text{original volume}}{\text{final volume}},$$

i.e. final pressure : original pressure

$$:: \frac{1}{\text{final volume}} : \frac{1}{\text{original volume}}.$$

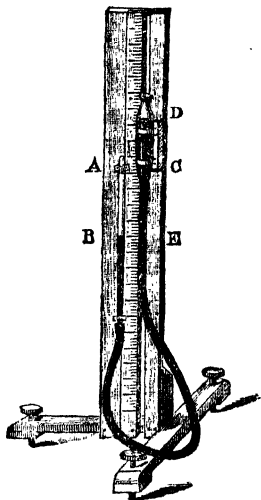
This proves the law for a diminution in the volume of the air.

295. Boyle's Law may also be verified by the following method, which is a modification of that of Art. 294, and is applicable to both increases and decreases of the volume of the air.

AB and *CD* are two glass tubes which are connected by flexible rubber tubing and are attached to a vertical stand. *AB* is closed at the top but *CD* is open. A vertical scale is fixed to the stand, and *CD* can move in a vertical direction parallel to this scale. The rubber tubing and the lower parts of the glass tubes are filled with mercury. The upper part of the tube *AB* is filled with air, and its pressure at any time is measured by $h + ED$, where *E* is at the same level as *B* and *h* is the height of the mercury barometer. Raise or lower the movable tube *CD*. Then in all cases it will be found that

$$AB \propto \frac{1}{h + ED},$$

$$\text{i.e. that volume} \propto \frac{1}{\text{pressure}}.$$



296. We have spoken as if Boyle's Law was perfectly accurate; until comparatively recent times it was supposed to be so. More careful experiments have shewn that it is not strictly accurate for all gases. It is however extremely near the truth for gases which are very hard to liquefy, such as air, oxygen, hydrogen, and nitrogen. Most gases are rather more compressible than Boyle's Law would imply.

A gas which accurately obeyed Boyle's Law would be called a **Perfect Gas**. The above-mentioned gases are nearly perfect gases.

297. Let p' be the original pressure, v' the original volume, and ρ' the original density of a given mass of gas.

When the volume of this gas has been altered, the temperature remaining constant, let p be the new pressure, v the new volume, and ρ the new density of the gas.

Boyle's Law states that

$$\frac{p}{p'} = \frac{v'}{v},$$

i.e.

$$p \cdot v = p' \cdot v' \dots\dots\dots(1).$$

Now $\rho \cdot v$ and $\rho' \cdot v'$ are each equal to the given mass of the gas which cannot be altered.

$$\therefore \rho \cdot v = \rho' \cdot v' \dots\dots\dots(2).$$

From (1) and (2), by division,

$$\frac{p}{\rho} = \frac{p'}{\rho'}.$$

Hence $\frac{p}{\rho}$ is always the same for a given gas. Let its value be denoted by k , so that $p = k\rho$.

Ex. Assuming the sp. gr. of air to be .0013 when the height of the mercury barometer is 30 inches, the sp. gr. of mercury to be 13.596, and the value of g to be 32.2, prove that the value of k , for foot-second units, is 841,906 nearly.

Find also the value for c.g.s. units, assuming that $g = 981$ and that the height of the mercury barometer is 76 cms.

$$p = \frac{30}{12} \times 13.596 \times g \times 62\frac{1}{2} \text{ poundals per square foot,}$$

and

$$\rho = .0013 \times 62\frac{1}{2} \text{ lbs.}$$

$$\therefore k = \frac{30}{12} \times 13.596 \times 32.2 \times \frac{1}{.0013} = 841,906 \text{ nearly.}$$

In c.g.s. units

$$p = 76 \times 13.596 \times 981 \text{ dynes per sq. cm.,}$$

and

$$\rho = .0013 \text{ gramme per cub. cm.}$$

$$\therefore k = \frac{76 \times 13.596 \times 981}{.0013} = 779,741,000 \text{ nearly.}$$

298. Ex. 1. The sp. gr. of mercury is 13.6 and the barometer stands at 30 ins. A bubble of gas, the volume of which is 1 cub. in. when it is at the bottom of a lake 170 ft. deep, rises to the surface. What will be its volume when it reaches the surface?

If w be the weight of a cub. ft. of the water, the pressure per sq. ft. at the bottom of the lake

$$= 170w + 13.6 \times 2\frac{1}{2}w = 204w.$$

Also the pressure at the top of the lake $= 13.6 \times 2\frac{1}{2}w = 34w$.

Hence, if x be the required volume, we have

$$x \times 34w = 1 \times 204w.$$

$$\therefore x = 6 \text{ cub. ins.}$$

Ex. 2. At what depth in water would a bubble of air sink, given that the weights of a cub. ft. of water and air are respectively 1000 and $1\frac{1}{4}$ ozs., and that the height of the water barometer is 34 ft.?

Let x be the depth at which the bubble would just float. Then, by Boyle's Law,

$$\begin{aligned}\frac{x+34}{34} &= \frac{\text{density of air at depth } x}{\text{atmospheric density}} \\ &= \frac{\text{density of water}}{\text{atmospheric density}}, \text{ since the bubble just floats,} \\ &= \frac{1000}{1\frac{1}{4}} = 800. \\ \therefore x &= 27,166 \text{ ft.} = \text{slightly greater than 5 miles.}\end{aligned}$$

Below this depth the bubble would sink; above this depth it would rise.

Ex. 3. 10 cub. cms. of air at atmospheric pressure are measured off. When introduced into the vacuum of a barometer they depress the mercury, which originally stood at 76 cms., and occupy a volume of 15 cub. cms. What is the final height of the barometer?

Let Π denote the atmospheric pressure. By Boyle's Law, we have

$$\frac{\text{final pressure of the air}}{\Pi} = \frac{\text{original volume}}{\text{final volume}} = \frac{10}{15} = \frac{2}{3}.$$

$$\therefore \text{final pressure of the air} = \frac{2}{3}\Pi.$$

The pressure above the column of mercury is now $\frac{2}{3}$ of atmospheric pressure, so that the length of the column of mercury is only $\frac{1}{3}$ of its original length and is therefore $25\frac{1}{3}$ cms.

Ex. 4. When the reading of the true barometer is 30 ins. the reading of a barometer, the tube of which contains a small quantity of air whose length is then $3\frac{1}{3}$ ins., is 28 ins. If the reading of the true barometer falls to 29 ins., prove that the reading of the faulty barometer will be $27\frac{1}{3}$ ins.

At an atmospheric pressure of 30 ins. of mercury, let x ins. be the length of the column of air. When the length is $3\frac{1}{3}$ ins. its pressure per square inch

$$= \frac{x}{3\frac{1}{3}} \times \text{atmospheric pressure} = \frac{x}{3\frac{1}{3}} \cdot w \cdot 30,$$

where w is the weight of a cubic inch of mercury.

Hence, for the equilibrium of the faulty barometer, we have

$$\begin{aligned}\frac{x}{3\frac{1}{3}} \cdot w \cdot 30 + w \cdot 28 &= \text{atmospheric pressure} = w \cdot 30. \\ \therefore x &= \frac{2}{9}.\end{aligned}$$

When the real barometer pressure is 29 ins., let the height of the faulty barometer be y ins., so that the pressure of the air per sq. inch

$$\begin{aligned}&= \frac{x \times 30}{31\frac{1}{3} - y} w, \text{ since the total length of the barometer tube} = 28 + 3\frac{1}{3} \\ &= 31\frac{1}{3} \text{ ins.}\end{aligned}$$

$$\therefore 29 = y + \frac{x}{31\frac{1}{3} - y} \times 30 = y + \frac{20}{34 - 3y}.$$

$$\therefore y = 27\frac{1}{3},$$

the other solution of this equation, viz. 33, being clearly inadmissible.

EXAMPLES. LIII

1. What is the sp. gr. of the air at standard pressure (760 mms. of mercury) when the sp. gr. of air at a pressure of 700 mms. of mercury, referred to water at 4° C. as standard, is found to be .00119?

2. When the height of the mercurial barometer changes from 29.45 ins. to 30.23 ins., what is the change in the weight of 1000 cub. ins. of air, assuming that 100 cub. ins. of air weigh 31 grains at the former pressure?

3. Assuming that 100 cub. ins. of air weigh 35 grains when the barometer stands at 30 ins., find the weight of the air that gets out of a room when the barometer falls from 30 ins. to 29 ins., the dimensions of the room being 30 ft. by 20 ft. by 15 ft.

4. When the water barometer is standing at 33 ft. a bubble at a depth of 10 ft. from the surface of water has a volume of 3 cub. ins. At what depth will its volume be 2 cub. ins.?

5. An air-bubble at the bottom of a pond, 10 ft. deep, has a volume of .00006 of a cub. in. Find what its volume becomes when it just reaches the surface, the barometer standing at 30 ins. and mercury being 13.6 times as heavy as water.

6. Assuming the height of the water barometer to be 34 ft., find to what depth an inverted tumbler must be submerged so that the volume of the air inside may be reduced to one-third of its original volume.

7. A cylindrical test tube is held in a vertical position and immersed mouth downwards in water. When the middle of the tube is at a depth of 32.75 ft. it is found that the water has risen halfway up the tube. Find the height of the water barometer.

8. A uniform tube, closed at the top and open at the bottom, is plunged into mercury, so that 25 cms. of its length are occupied by gas at an atmospheric pressure of 76 cms. of mercury; the tube is now raised till the gas occupies 50 cms.; by how much has it been raised?

9. What are the uses of the small hole which is made in the lid of a teapot and of the vent-peg of a beer barrel?

10. A hollow closed cylinder, of length 2 ft., is full of air at the atmospheric pressure of 15 lbs. per square inch when a piston is 12 ins. from the base of the cylinder; more air is forced in through a hole in

the base of the cylinder till there is altogether three times as much air in the cylinder as at first; if the piston be now allowed to rise 4 ins., what is the pressure of the air on each side of the piston?

Through how many inches must the piston move from its original position to be again in equilibrium?

11. A balloon half filled with coal-gas just floats in the air when the mercury barometer stands at 30 ins. What will happen if the barometer sinks to 28 ins.?

What would happen if the balloon had been quite full of gas at the higher pressure?

12. Why does a small quantity of air introduced into the upper part of a barometer tube depress the mercury considerably whilst a small portion of iron floating on the mercury does not depress it?

13. A barometer stands at 30 ins. The vacuum above the mercury is perfect, and the area of the cross-section of the tube is a quarter of a sq. in. If a quarter of a cub. in. of the external air be allowed to get into the barometer, and the mercury then falls 4 ins., what was the volume of the original vacuum?

14. A bubble of air having a volume of 1 cub. in. at a pressure of 30 ins. of mercury escapes up a barometer tube, whose cross-section is 1 sq. in. and whose vacuum is 1 in. long. How much will the mercury descend?

15. The top of a uniform barometer tube is 33 ins. above the mercury in the tank, but on account of air in the tube the barometer registers 28.6 ins. when the atmospheric pressure is equivalent to that of 29 ins. of mercury. What will be the true height of the barometer when the height registered is 29.48 ins.?

16. The top of a uniform barometer tube is 36 ins. above the surface of the mercury in the tank. In consequence of the pressure of air above the mercury the barometer reads 27 ins. when it should read 28.5 ins. What will be the true height when the reading of the barometer is 30 ins.?

17. The readings of a true barometer and of a barometer which contains a small quantity of air in the upper portion of the tube are respectively 30 and 28 ins. When both barometers are placed under the receiver of an air-pump from which the air is partially exhausted, the readings are observed to be 15 and 14.6 ins. respectively.

Prove that the length of the tube of the faulty barometer measured from the surface of the mercury in the basin is 31.35 ins.

18. The two limbs of a Mariotte's tube are graduated in inches. The mercury in the shorter tube stands at the graduation 4, and 5 ins. of air are enclosed above it. The mercury in the other limb stands at the graduation 38, and the barometer at the time indicates a pressure of 29.5 ins. Find to what pressure the 5 ins. of air are subjected and also the length of the tube they would occupy under barometric pressure alone.

19. A gas-holder consists of a cylindrical vessel inverted over water. Its diameter is $2\frac{1}{2}$ ft. and its weight 60 lbs. Find what part of the weight of the cylinder must be counterpoised to make it supply gas at a pressure equivalent to that of 1 in. of water.

20. A pint bottle containing atmospheric air just floats in water when it is weighted with 5 ozs. The weight is then removed, and the bottle immersed neck downwards and gently pressed down.

Show that it will just float freely when the level of the water inside the bottle is 11 ft. below the surface, and will sink if lowered farther, and rise if raised higher. The water barometer stands at 33 ft., and a pint of water weighs 20 ozs.

21. A closed air-tight cylinder, of height $2a$, is half full of water and half full of air at atmospheric pressure, which is equal to that of a column, of height h , of the water. Water is introduced without letting the air escape so as to fill an additional height k of the cylinder, and the pressure on the base is thereby doubled. Prove that

$$k = a + h - \sqrt{ah + h^2}.$$

22. In a vertical cylinder, the horizontal section of which is a square of side 1 ft., is fitted a weightless piston. Initially the air below the piston occupies a space 7 ft. in length and is at the same pressure as the external air. 6 cub. ft. of water are taken and two-thirds of a cub. ft. of iron. If the iron be placed on the piston, it sinks 1 ft. If the water also be then poured on it, it sinks through $\frac{36}{41}$ ft. Find the sp. gr. of iron, and the height of the water barometer.

299. *Relation between the pressure, temperature, and density of a gas.*

It can be shewn experimentally that a given mass of gas, for each increase of 1°C. in its temperature, has its volume increased (provided its pressure remain constant) by an amount which is equal to $\cdot 003665$ times ($= \frac{1}{273}$ nearly) its volume at 0°C.

Thus, if V_0 be the volume of the given mass of gas at temperature 0°C. and α stand for $\cdot 003665$, the increase in volume for each degree Centigrade of temperature is αV_0 . Hence the increase for $t^\circ \text{C.}$ is $\alpha V_0 \cdot t$, so that if V be the volume of this air at temperature $t^\circ \text{C.}$, then

$$V = V_0 + \alpha V_0 t = V_0 (1 + \alpha t).$$

If ρ and ρ_0 be the respective densities at the temperatures $t^\circ \text{C.}$ and 0°C. , then, since

$$\rho V = \rho_0 V_0,$$

we have
$$\frac{\rho_0}{\rho} = \frac{V}{V_0} = 1 + \alpha t.$$

$$\therefore \rho_0 = \rho (1 + \alpha t).$$

The above law is sometimes known as **Gay-Lussac's**, and sometimes as **Charles'**.

300. A relation similar to that of the previous article holds for all gases. For those approximating to perfect gases α is very nearly the same quantity.

If the temperature be measured by the Fahrenheit thermometer, instead of the Centigrade, the value of α is $\frac{5}{9} \times \frac{1}{273}$ nearly [for 180 degrees on the Fahrenheit scale equal 100 degrees on the Centigrade scale, i.e. $1^\circ \text{F.} = \frac{5}{9}^\circ \text{C.}$].

301. Ex. 1. If the volume of a certain quantity of air at a temperature of 10°C. be 300 cub. cms., what will be its volume (at the same pressure) when its temperature is 20°C. ?

If V_0 be its volume at 0°C. , then

$$300 = V_0 + 10 \cdot \frac{1}{273} \cdot V_0 = \frac{283}{273} V_0.$$

$$\therefore V_0 = \frac{273}{283} \times 300.$$

Hence the volume at $20^\circ \text{C.} = V_0 + 20 \cdot \frac{1}{273} \cdot V_0$

$$= \frac{293}{273} V_0 = \frac{293}{283} \times 300 = 310 \frac{170}{283} \text{ cub. cms.}$$

Ex. 2. The volume of a certain quantity of gas at 15°C. is 400 cub. cms.; if the pressure be unaltered, at what temperature will its volume be 500 cub. cms.?

Let t be the required temperature. Then

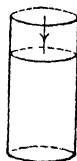
$$\frac{500}{400} = \frac{\text{volume at temp. } t^\circ \text{C.}}{\text{volume at temp. } 15^\circ \text{C.}} = \frac{1 + \frac{t}{273}}{1 + \frac{15}{273}} = \frac{273 + t}{288}.$$

$$\therefore t = 87^\circ \text{C.}$$

302. Suppose the gas at a temperature 0°C. to be confined in a cylinder, and to support a piston of such a weight that the pressure of the gas is p , and let the density of the gas be ρ_0 , so that

$$p = k\rho_0 \dots\dots\dots(1).$$

Let heat be applied to the cylinder till the temperature of the gas is raised to $t^\circ \text{C.}$, and let the density then be ρ .



By Charles' Law we then have

$$\rho_0 = \rho (1 + \alpha t) \dots\dots\dots(2).$$

From (1) and (2), we have

$$p = k\rho (1 + \alpha t),$$

giving the relation between the pressure, density, and temperature of the gas.

303. Absolute temperature. If a gas were continually cooled till its temperature was far below 0°C . and if it did not liquefy and continued to obey Charles' and Boyle's Laws, the pressure would be zero for a temperature t , such that

$$1 + \alpha t = 0,$$

$$\text{i.e. when} \quad t = -\frac{1}{\alpha} = -273.$$

This temperature -273° is called the absolute zero of the Air Thermometer, and the temperature of the gas measured from this zero is called the absolute temperature. The absolute temperature is generally denoted by T , so that

$$T = \frac{1}{\alpha} + t.$$

Hence

$$p = k\rho (1 + \alpha t) = k\rho \alpha \left(\frac{1}{\alpha} + t \right) = k\rho \alpha T.$$

Therefore, if V be the volume of a certain quantity of gas, we have

$$\frac{p \cdot V}{T} = k\alpha \cdot [V \cdot \rho] = k\alpha \times \text{mass of the gas} = \text{a constant}.$$

Hence the product of the pressure and volume of any given mass of gas is proportional to its absolute temperature.

Ex. The radius of a sphere containing air is doubled, and the temperature raised from 0°C . to 91°C . Prove that the pressure of the air is reduced to one-sixth of its original value, the coefficient of expansion per 1°C . being $\frac{1}{273}$.

Let p be the original and p' the final pressure, ρ the original and ρ' the final density.

Since the radius of the sphere is doubled, the final volume is eight times the original volume.

$$\begin{aligned}\therefore \rho' &= \frac{1}{8}\rho. \\ \therefore \frac{p'}{p} &= \frac{k\rho'(1 + a \cdot 91)}{k\rho} = \frac{1}{8} \left[1 + \frac{91}{273} \right] \\ &= \frac{1}{8} \times \frac{4}{3} = \frac{1}{6}.\end{aligned}$$

EXAMPLES. LIV

[In the following examples take a as $\frac{1}{273}$.]

1. If a quantity of gas under a pressure of 57 ins. of mercury and at a temperature of 69°C . occupy a volume of 9 cub. ins., what volume will it occupy under a pressure of 51 ins. of mercury and at a temperature of 16°C .?

2. A mass of air at a temperature of 39°C . and a pressure of 32 ins. of mercury occupies a volume of 15 cub. ins. What volume will it occupy at a temperature of 78°C . under a pressure of 54 ins. of mercury?

3. At the sea-level the barometer stands at 750 mm. and the temperature is 7°C ., while on the top of a mountain it stands at 400 mm. and the temperature is 13°C .; compare the weights of a cubic metre of air at the two places.

4. If a quantity of gas occupies 228 c.c. at a temperature of 27°C . and a pressure of 750 mm., what is its temperature when its volume is 216 c.c. and its pressure is 760 mm.?

5. A cylinder contains two gases which are separated from each other by a movable piston. The gases are both at 0°C ., and the volume of one gas is double that of the other. If the temperature of the first be raised t° , prove that the piston will move through a space $\frac{2lat}{9 + 6at}$, where l is the length of the cylinder, and a is the coefficient of expansion per 1°C .

MISCELLANEOUS EXAMPLES

1. Find the resultant of two forces, equal to the weights of 13 and 14 lbs. respectively, acting at an obtuse angle whose sine is $\frac{1}{13}$.
2. Resolve a force of 100 lbs. weight into two equal forces acting at an angle of 60° .
3. $ABCD$ is a square; forces of 1 lb. wt., 6 lbs. wt., and 9 lbs. wt. act in the directions AB , AC , and AD respectively; find the magnitude of their resultant correct to two places of decimals.
4. The resultant of two forces, acting at an angle of 120° , is perpendicular to the smaller component. The greater component is equal to 100 lbs. weight; find the other component and the resultant.
5. A stiff pole, 12 feet long, sticks horizontally out from a vertical wall. It would break if a weight of 28 lbs. were hung at the end. How far out along the pole may a boy who weighs 8 stone venture with safety?
6. A rod, weighing 4 ounces and of length 1 yard, is placed on a table so that one-third of its length projects over the edge. Find the greatest weight which can be attached by a string to the end of the rod without causing it to topple over.
7. A uniform beam, of weight 30 lbs., rests with its lower end on the ground, the upper end being attached to a weight by means of a horizontal string passing over a small pulley. If the beam be inclined at 60° to the vertical, prove that the resultant reaction at the lower end is nearly 40 lbs. wt., and that the weight attached to the string is nearly 26 lbs. wt.
8. Find the centre of parallel forces which are equal respectively to 1, 2, 3, 4, 5, and 6 lbs. weight, the points of application of the forces being at distances 1, 2, 3, 4, 5, and 6 inches respectively measured from a given point A along a given line AB .
9. The angle B of a triangle ABC is a right angle, AB being 8 inches and BC 11 inches in length; at A , B , and C are placed particles whose weights are 4, 5, and 6 lbs. respectively; find the distance of their centre of gravity from A .
10. On the side AB of an equilateral triangle, and on the side remote from C , is described a rectangle whose height is one-half of AB ; prove that the centre of gravity of the whole figure thus formed is the middle point of AB .
11. From a regular hexagon one of the equilateral triangles, with its vertex at the centre and a side for base, is cut away. Find the centre of gravity of the remainder.
12. The thrust on the fulcrum when two weights are suspended in equilibrium at the ends of a straight lever, 12 inches long, is 20 lbs. wt., and the ratio of the distances of the fulcrum from the ends is 3 : 2. Find the weights.

13. A straight lever, of length 5 feet and weight 10 lbs., has its fulcrum at one end, and weights of 3 and 6 lbs. are fastened to it at distances of 1 foot and 3 feet from the fulcrum; it is kept horizontal by a force at its other end; find the reaction at the fulcrum.

14. Find the relation between the effort P and the weight W in a system of 5 movable pulleys in which each pulley hangs by a separate string, the weight of each pulley being P .

15. In a system of 5 weightless pulleys in which each string is attached to a weightless bar from which the weight hangs, if the strings be successively 1 inch apart, find to what point of the bar the weight must be attached, so that the bar may be always horizontal.

16. A body, of mass 5 lbs., rests on a smooth plane which is inclined at 30° to the horizon, and is acted on by a force equal to the weight of 2 lbs. acting parallel to the plane and upwards, and by a force equal to P lbs. weight acting at an angle of 30° to the plane. Find the value of P if the body be in equilibrium.

17. If one scale of an accurate balance be removed and no mass be placed in the other scale, prove that the inclination of the beam to the horizon is the angle whose tangent is $\frac{Sa}{W'k + Sh}$, where $2a$ is the length of the beam, h and k are respectively the distances of the point of suspension from the beam and the centre of gravity of the balance, and S and W' are respectively the weights of the scale-pan and the remainder of the balance.

18. If the distance of the centre of gravity of the beam of a common steelyard from its fulcrum be 2 inches, the movable weight 4 ozs., and the weight of the beam 2 lbs., find the distance of the zero of graduation from the centre of gravity. Also, if the distance between the fulcrum and the end at which the scale-pan is attached be 4 inches, find the distance between successive graduations.

19. If the circumference of a smooth screw is 20 inches, and the distance between successive threads $\cdot 75$ inch, find the mechanical advantage.

20. The height of a rough plane is to its base as 3 to 4, and it is found that a body is just supported on it by a horizontal force equal to half the weight of the body; find the coefficient of friction between the body and the plane.

21. A cylindrical shaft has to be sunk to a depth of 100 fathoms through chalk whose density is 2.3 times that of water; the diameter of the shaft being 10 feet, what must be the H.P. of the engine that can lift out the material in 12 working days of 8 hours each? [A cubic foot of water weighs $62\frac{1}{2}$ lbs.]

22. Two forces act at an angle of 120° . The greater is represented by 80, and the resultant is at right angles to the less. Find the latter.

23. If one of two forces be double the other and the resultant be equal to the greater force, find the angle between the forces.

24. Two forces acting on a particle are at right angles and are balanced by a third force making an angle of 150° with one of them. The greater of the two forces being 3 lbs. weight, what must be the values of the other two?

25. The resultant of two forces P and Q acting at an angle θ is equal to $(2m+1)\sqrt{P^2+Q^2}$; when they act at an angle $90^\circ-\theta$, the resultant is $(2m-1)\sqrt{P^2+Q^2}$; prove that

$$\tan \theta = \frac{m-1}{m+1}.$$

26. Three equal strings, of no sensible weight, are knotted together to form an equilateral triangle ABC , and a weight W is suspended from A . If the triangle and weight be supported, with BC horizontal, by means of two strings at B and C , each at an angle of 135° with BC , shew that the tension in BC is

$$\frac{W}{6}(3-\sqrt{3}).$$

27. A uniform bar, of length $7\frac{1}{2}$ feet and weight 17 lbs., rests on a horizontal table with one end projecting $2\frac{1}{2}$ feet over the edge; find the greatest weight that can be attached to its end, without making the bar topple over.

28. A steelyard by use loses $\frac{1}{10}$ th of its weight, its centre of gravity remaining unaltered; shew how to correct its graduations.

29. The diameter of the circular piston of a steam engine is 60 inches and it makes 11 strokes per minute, the length of each stroke being 8 feet, the mean pressure per square inch on the piston being 15 lbs. wt., and the efficiency of the engine .65. Find the number of cubic feet of water that it will raise per hour from a well whose depth is 300 feet, on the supposition that no work is wasted.

30. The diameter of the piston of an engine is 80 inches, the mean pressure of steam 12 lbs. per square inch, the length of the stroke 10 feet, and the number of double strokes per minute 11. The engine is found to raise $42\frac{1}{2}$ cub. ft. of water per minute from a depth of 500 fathoms. Shew that its efficiency is .6 nearly.

31. The radii of a wheel and axle are 4 feet and 6 inches. If a force of 56 lbs. wt. is required to overcome a resistance of 200 lbs. wt., what is the efficiency of the machine?

32. In some experiments with a block and tackle (second system of pulleys), in which the velocity ratio was 4, the weights lifted were 10, 80, and 160 lbs., and the corresponding values of the efforts were 23, 58, and 85 lbs. Find the efficiency in each case.

33. With a certain machine it is found that, with efforts equal to 12 and 7.5 lbs. wt. respectively, resistances equal to 700 and 300 lbs. wt. are overcome; assuming that $P=a+bW$, find the values of a and b .

34. In some experiments with a screw-jack the values of the load W were 150, 180, 210, 240, and 270 lbs. wt. and the corresponding values of the effort P were found to be 20·9, 22·7, 25·75, 28·4, and 31·4 lbs. wt.; plot the results on squared paper and, assuming that $P = a + bW$, find the approximate values of a and b .

35. The handles of a wheelbarrow are 4 feet, and its legs 3 feet, from the axle of the wheel. A force of 40 lbs. weight applied to the axle will just raise the wheel off the ground, and a force of 30 lbs. wt. applied at the end of each handle will just raise the legs off the ground. Find the weight of the barrow, and the distance of its centre of gravity from the axle of the wheel.

36. A uniform lamina in the shape of a right-angled triangle, and such that one of the sides containing the right angle is three times the other, is suspended by a string attached to the right angle; in the position of equilibrium, shew that the hypotenuse is inclined at an angle $\sin^{-1} \frac{3}{5}$ to the vertical.

37. Find the vertical angle of a cone in order that the centre of gravity of its whole surface, including its plane base, may coincide with the centre of gravity of its volume. [See Appendix II, Arts. 8 and 9.]

38. A cylinder and a cone have their bases joined together, the bases being of the same size; find the ratio of the height of the cone to the height of the cylinder so that the common centre of gravity may be at the centre of the common base.

39. A triangular lamina ABC is acted upon by three forces: P from A to B , $2P$ from B to C , and $3P$ from A to C . Shew that the resultant of these three forces passes through the centre of the circle inscribed in the triangle ABC .

40. A fishing-rod consists of three parts, each 3 ft. 6 ins. in length. The butt weighs 10 ozs. and balances about a point distant 1 foot from its lower end. The middle piece weighs 4 ozs. and balances about a point 1 ft. 3 ins. from its lower end. The top weighs 2 ozs. and balances about a point 1 ft. 6 ins. from its lower end. When fitted together each piece overlaps the next by 2 ins. Find the position of the centre of gravity of the whole rod when put together.

41. A piece of paper in the form of a square $ABCD$ is folded so that the vertex C coincides with the middle point O of the diagonal BD ; shew that the new centre of gravity divides OA in the ratio 1 : 23.

42. A triangular piece of paper, whose sides are each 1 foot in length, is folded so that one vertex A is made to coincide with the middle point of the opposite side. Shew that the distance of the centre of gravity of the folded paper from the original position of A is $\frac{9\sqrt{3}}{2}$ ($= 7\cdot79$) inches.

43. A steamer with engines of 25,000 H.P. goes at the rate of 20 miles per hour. What is the resistance of the water to its motion?

44. A motor runs at 1500 revolutions per minute, and exerts on a pulley a couple equal to 5 foot-pound units. At what rate is it doing work?

45. In a block and tackle (the second system of pulleys) the velocity ratio is 8:1. The friction is such that only 55% of the force applied can be usefully employed. Find what force will raise 5 cwt. by its use.

46. The velocity ratio of a machine is 40, and it is found that efforts of 10, 16, and 22 lbs. wt. are necessary to lift loads of 200, 400, and 600 lbs. Find the effort required to lift half a ton, and the percentage efficiency of the machine for each load.

47. Find the velocity ratio, the mechanical advantage, and the efficiency of a screw-jack in which the pitch of the screw is $\frac{1}{4}$ inch, the length of the arm is 20 inches, and the tangential force at the end of the arm required to lift 1 ton is 12 lbs. wt.

48. A particle falls freely from the top of a tower, and during the last second of its motion it falls $\frac{5}{8}$ ths of the whole height; what is the height of the tower?

49. A balloon ascends with a uniform acceleration of 4 ft.-sec. units; at the end of half a minute a body is released from it; find the time that elapses before the body reaches the ground.

50. A tower is 288 feet high; one body is dropped from the top of the tower and at the same instant another is projected vertically upwards from the bottom, and they meet half-way up; find the initial velocity of the projected body and its velocity when it meets the descending body.

51. A body is dropped from the top of a given tower, and at the same instant a second body is projected from the foot of the tower, in the same vertical line, with a velocity which would be just sufficient to take it to the same height as the first; find where they will meet.

52. A mass of 350 grammes is placed on a smooth table at a distance of 245.25 cms. from its edge and is connected by a light string passing over the edge with a mass of 50 grammes hanging freely; what time will elapse before the hanging mass leaves the table?

53. Two scale-pans, each of mass 5 ozs, are connected by a light inextensible string which passes over a pulley. If a mass of 5 ozs be placed in one pan and a mass of 10 ozs in the other, find the forces exerted by the masses on the pans.

54. Shew that a train going at 11 miles per hour will be brought to rest in about 84 yards if the engine has brakes, if they press on the wheels with a force equal to $\frac{1}{10}$ of the weight of the train, the coefficient of friction be $\frac{1}{10}$.

55. A train of mass 50 tons is moving at the rate of 30 miles per hour when the steam is shut off and the brake being applied.

applied to the brake-van, the train is stopped in a quarter of a mile. Find the mass of the brake-van, taking the coefficient of friction between the wheels and rails to be one-sixth, and supposing the unlocked wheels to roll without sliding.

56. A mass of half a ton, moving with a velocity of 800 feet per second, strikes a fixed target and is brought to rest in a hundredth part of a second. Find the impulse of the blow on the target and, supposing the resistance to be uniform throughout the time taken to bring the body to rest, find the distance through which it penetrates.

57. If a bicyclist always works with $\frac{1}{10}$ H.P. and goes 12 miles per hour on the level, shew that the resistance of the road is 3.125 lbs. wt. If the mass of the machine and its rider be 12 stone, shew that up an incline of 1 in 50 the speed will be reduced to about 5.8 miles per hour.

58. A particle is dropped from a height h , and after falling $\frac{2}{3}$ rds of that distance passes a particle which was projected upwards at the instant when the first was dropped. Find to what height the latter will attain.

59. A body begins to slide down a smooth inclined plane from rest at the top, and at the same instant another body is projected upwards from the foot of the plane with such a velocity that they meet halfway up the plane; find the velocity of projection and determine the velocity of each when they meet.

60. Two equal masses, attached by an inextensible weightless thread which passes over a light pulley, hang in equilibrium. Shew that the tension of the thread is unaltered when $\frac{1}{n}$ th of its mass is added to one, and $\frac{1}{n+2}$ th of its mass is removed from the other.

61. A slip-carriage was detached from a train and brought to rest in n minutes, during which it described a distance of s feet. Assuming the retardation to be uniform, find the velocity with which the train was moving when the carriage was slipped.

62. A ship sailing south with a velocity of u miles an hour, and another ship, which is steaming north with a velocity of v miles an hour, always appears to be in a direction θ from the line of sight nearer. Find the direction of the motion of the second vessel.

63. Two masses of m and M respectively are connected by a string passing over a pulley; at the end of 3 seconds the larger mass imparts a velocity v to the smaller mass, and the inelastic horizontal plane; shew that the system will be at rest at the end of $2\frac{1}{2}$ seconds more.

64. On a certain day an inch of rain fell in 3 hours; assuming that the drops are very small and that the terminal velocity is 10 feet per second, find the impulsive pressure in tons per square

mile consequent on their being reduced to rest, it being given that the mass of a cubic foot of water is 1000 ounces and that the rain was uniform and continuous.

65. A bullet, moving with a velocity of 1200 ft. per sec., has this velocity reduced to one-half after penetrating 1 inch into a target. Assuming the resistance to be uniform, how far will it penetrate before its velocity is destroyed?

66. A shot of mass m is fired from a gun of mass M with velocity u relative to the gun; shew that the actual velocities of the shot and gun are $\frac{Mu}{M+m}$ and $\frac{mu}{M+m}$ respectively, and that their kinetic energies are inversely proportional to their masses.

67. A body is projected upwards with velocity u , and t seconds afterwards another body is similarly projected with the same velocity; find when and where they will meet.

68. A weightless string, of length a , with masses m and $3m$ attached to its ends, is placed on a smooth horizontal table perpendicular to an edge with the mass m just over the edge. If the height of the table above the inelastic floor be also a , shew that the mass $3m$ will strike the floor at a distance a from the mass m .

69. A locomotive engine draws a load of m lbs. up an incline of inclination α to the horizon, the coefficient of friction being μ . If, starting from rest and moving with uniform acceleration, it acquires a velocity v in t seconds, shew that the average horse-power at which the engine has worked is $\frac{mv}{1100} \left[\frac{v}{gt} + \mu \cos \alpha + \sin \alpha \right]$.

70. A body is thrown up in a lift with a velocity u relative to the lift, and the time of flight is found to be t . Shew that the lift is moving upwards with an acceleration $\frac{2u - tg}{t}$.

71. A mass of 4 cwt. falls from a height of 10 feet upon an inelastic pile of mass 12 cwt.; supposing the mean resistance of the ground to penetration by the pile to be $1\frac{1}{2}$ tons weight, determine the distance through which the pile is driven at each blow, and the time it takes to travel this distance.

Find also what fraction of the energy is dissipated at each blow.

72. The mass of a rider and his bicycle is 180 lbs.; the machine is running freely down an incline of 1 in 60 at a uniform rate of 8 miles per hour; shew that to go at the same rate up an incline of 1 in 100 he must work at the rate of .1024 H.P.

73. A particle is projected at an angle of elevation $\sin^{-1} \frac{4}{5}$, and its range on the horizontal plane is 4 miles; find the velocity of projection, and the velocity at the highest point of its path.

74. Find the angle of projection of a particle when its range on a horizontal plane is (i) 4, (ii) $4\sqrt{3}$ times the greatest height attained.

75. The height of an inclined plane is $\frac{3}{4}$ ths of its length; a body is projected up the plane from the bottom with a velocity of 50 ft. per sec. and slides down again; find the distance the body goes up the plane, and the time that elapses before it again arrives at the starting-point.

76. Two particles start from rest at the same instant with velocities u and v respectively; the motion of the first is uniformly retarded and that of the second is uniform; by the time the first comes to rest, shew that the distances traversed are as $u : 2v$.

77. A train takes 3 minutes to go between two stations, one mile apart, starting from and finishing at rest. It is uniformly accelerated for the first 30 seconds, and uniformly retarded for the last 15 seconds, the speed being constant for the remaining time. Calculate the values of the acceleration, retardation, and uniform velocity.

78. A pump is employed to raise water from a depth of 150 feet, and to deliver it with a velocity of 20 ft. per sec. Shew that 96 per cent. of the work done by the pump is employed in raising the water, and 4 per cent. in imparting to it its velocity at efflux.

79. If a motor-car going at 100 kilometres per hour can be stopped in 200 metres, shew that the brakes can hold the car on an incline of 1 in 5, and find the time that the motor-car takes in stopping.

80. A motor-car, of total weight 1 ton, ascends a hill which rises 3 feet in every 44 yards at a speed of 20 miles per hour. If the resistance, due to friction and the wind, is equal to 84 lbs. weight, shew that the horse-power exerted by the motor is approximately 7.195.

81. A light car, whose mass is 15 cwt., starts from rest at the foot of an incline of 1 in 25, and after travelling half a mile has attained a speed of 20 miles per hour. If the resistance due to friction etc. is 60 lbs. weight, find the force which must act on the car, if it is constant throughout the motion.

82. A mass of 30 lbs. is attached to a spring balance in a lift moving at such a rate that the balance indicates a weight of (i) 25 lbs., and (ii) 32 lbs. Find the acceleration of the lift in the two cases.

83. A body, of mass 5 lbs., is moving northwards with a velocity of 10 feet per second. After a constant force has acted on it for 5 seconds, it is found to be moving in a north-easterly direction with a velocity of $10\sqrt{2}$ feet per second. Find the magnitude and direction of this constant force.

What is the gain of kinetic energy?

84. Find the angle of projection of a particle when its range on a horizontal plane is equal to the distance through which the particle would have to fall in order to acquire a velocity equal to its velocity of projection.

85. Shew that four times the square of the number of seconds in the time of flight in the range on a horizontal plane equals the height in feet of the highest point of the trajectory.

86. A gun, weighing 6 tons, fires a 12 lb. shot horizontally with a velocity of 2000 feet per second. Find the kinetic energy of the shot and of the gun in its recoil. What average force will be required to stop (1) the shot, and (2) the gun, in a distance of 1 foot?

87. A pile-driver, weighing 400 lbs., falls through 9 feet upon a pile weighing 80 lbs., and at each impact the pile is driven in 10 inches. If the resistance of the earth to the motion of the pile is assumed to be constant, shew that it is $1\frac{23}{8}$ tons weight.

88. Three small bodies, of masses 4, 5, and 6 ounces respectively, lie in order in a straight line on a large smooth table, the distances between consecutive bodies being 6 inches. Two slack strings, each 2 feet in length, connect the first with the second and the second with the third. The third body is projected with a speed of 15 feet per second directly away from the other two. Find the time that elapses before the first begins to move and the speed with which it starts. Find also the loss in energy.

89. There are two chambers, one of which is dropping freely under gravity down a pit, whilst the other descends with uniform velocity. A man in each chamber during the descent lets go a stone which he has been holding in his hand. What will be the motion of the stone in each case?

90. Plot a curve shewing the relation between the displacement and the time, during a period of 8 seconds, of a stone thrown vertically upwards with a velocity of 112 feet per second. At what time must another stone, originally situated at the highest point to which the first stone rises, begin falling in order to meet it halfway up?

91. A boat is moored at a place where the current of the river is flowing eastwards at the rate of $1\frac{1}{2}$ miles per hour. Two buoys are also moored, each 176 feet from the boat, one due north and the other due east of it. Two swimmers, each able to swim at $2\frac{1}{2}$ miles per hour in still water, start from the boat and swim one to each buoy and back again to the boat. Which will reach the boat again first and by how much? Which will turn first at his buoy?

92. A person on an express train moving at 60 miles per hour wishes to hit a stationary object which is situated 100 yards off in a line through the marksman at right angles to the line of motion of the train. If his bullet moves at the rate of 1200 ft. per sec., find how much to one side of the object he should aim.

93. A ship is sailing due east with a velocity of 20 miles per hour. A submarine, one mile due south of the ship, fires a torpedo at the rate of 25 miles per hour. In what direction must the torpedo be fired in order that it may strike the ship, and what time will elapse before it strikes?

94. A ship is sailing due south at 6 miles per hour, and a second ship is sailing due east at 10 miles per hour. Find, graphically or otherwise, the velocity of the second ship with respect to the first. If

initially the first ship was 5 miles due east of the second ship, find their nearest distance from one another.

95. A body A is moving north with a velocity of 10 miles per hour, and its velocity relative to another body B appears to be 15 miles per hour in a north-easterly direction. Find the velocity of B in magnitude and direction.

96. A ship is steaming westwards with a velocity of 15 miles per hour. If, to a passenger on board, the wind appears to blow from the south-west with a velocity of 21.2 miles per hour, find, graphically or otherwise, the direction and velocity of the wind relative to the sea.

97. The smoke from a steamer, moving at 12 knots due south, appears to an observer on the deck to be moving due east. The steamer then comes to rest, and the smoke moves in a south-easterly direction. What was the velocity of the wind?

98. A boat crosses from a point on one bank of a river to a point directly opposite on the other bank. If the velocity of the stream is 3 miles per hour and that of the boat is 10 miles per hour, find, graphically or otherwise, the direction in which the boat must be steered, and, if the river is a quarter of a mile wide, find the time it takes to cross.

99. A boat is rowed at the rate of 4 miles per hour across a stream, half a mile wide, which is flowing at the rate of 3 miles per hour. Find the shortest time for the crossing, and the direction of the crossing.

If the boat is to arrive at the point of the other bank which is exactly opposite to its starting-point, find the direction in which the boat must be rowed and the time of this crossing.

100. Two motor-cars are travelling along two roads which cross one another at right angles at A . One is 500 yards from A , and is travelling towards it at 21 miles an hour. The other is 700 yards from A , and is travelling towards it at 28 miles an hour. Find the velocity of the first car relative to the other, and prove that the distance between the cars will never be less than 20 yards.

101. A jet of water, two inches in diameter, impinges at right angles on a vertical plate. Find the force necessary to keep the plate in position if the velocity of the water in the jet is 20 feet per second, the mass of a cubic foot of water being $62\frac{1}{2}$ lbs., and the velocity of the water after striking the plate being neglected.

✓ 102. The sp. gr. of ice is .92 and that of sea-water is 1.025. What depth of water will be required to float a cubical ice-berg of side 100 yards?

✓ 103. A piece of wood, weighing a kilogramme, floats in water with $\frac{3}{8}$ ths of its volume immersed; find the density and the volume of the wood.

✓ 104. A piece of wood, of sp. gr. $\frac{2}{3}$, is floating in oil, of sp. gr. .84; what fraction of its volume is immersed?

105. A wide-mouthed bottle, full of air, is closed by a well-ground glass stopper, 5 cms. in diameter, the barometer standing at 772 mm. ; what weight must the stopper have in order that it may be just lifted if the barometer goes down to 730 mm., the temperature remaining unaltered? [1 c.c. of mercury weighs 13.6 grammes.]

106. In the experiment of Archimedes, Hiero's crown, and masses of gold and silver each equal in weight to the crown, were weighed in water in succession. The crown lost $\frac{1}{14}$ th of its weight, the gold $\frac{4}{77}$, and the silver $\frac{2}{21}$. In what proportion by weight were gold and silver mixed in the crown?

107. A closed cubical vessel, with sides one inch in thickness, is made of material whose sp. gr. is $2\frac{3}{4}$. If the vessel can float in water, shew that its internal volume must be at least 1000 cubic inches.

108. A body is composed of a hemisphere, of radius r , with a cone, of height $2r$, on the same plane base. In a liquid of density ρ_1 it floats with the whole body just immersed, and in a liquid of density ρ_2 with the hemisphere just immersed; prove that $\rho_2 = 2\rho_1$.

109. A cylinder, loaded so as to float vertically and weighing 2 grammes altogether, just sinks in water when half a gramme extra is put on its top; otherwise it protrudes 7 cms. above the surface. What length will appear above the surface of a liquid whose density is five times that of water, if the cylinder be set floating in it without the extra load?

110. An empty balloon, with its car and appendages, weighs in air 1200 lbs. If a cubic foot of air weighs $1\frac{1}{4}$ ozs., how many cubic feet of gas of sp. gr. .52 times that of air must be introduced before it begins to ascend?

111. A glass tumbler weighs 8 ozs. ; its external radius is $1\frac{1}{2}$ inches and its height is $4\frac{1}{2}$ inches; if it be allowed to float in water with its axis vertical, find what additional weight must be placed in it to make it sink.

112. A rod of cork 8 ins. long and a rod of lignum vitæ 4 ins. long are joined together to form a straight rod, one foot long, of uniform section. When the rod floats in water, it is found that it can rest with part of the cork above the water and its axis inclined at any angle to the vertical. If the sp. gr. of cork be .24, find that of lignum vitæ.

113. A piece of wood weighs 6 lbs. in air; a piece of lead which weighs 12 lbs. in water is fastened to it, and the two together weigh 10 lbs. in water; what is the sp. gr. of the wood?

114. A body, of sp. gr. ρ , floats half immersed in a liquid, but is three-quarters immersed in a mixture of equal volumes of the liquid and water. Neglecting the atmospheric pressure, find ρ .

115. A retort, of 3 litres capacity, and with its open end submerged 3.4 cms. below the surface of water in a trough, is seen to be completely full of air on a certain day. Next day the mercury barometer is observed

to have fallen 2 cms. to 74 cms. There being no change of temperature, how much of the air originally in the retort has by this time bubbled out? [Sp. gr. of mercury = 13·6.]

116. The length of a barometer tube is 80 inches and its diameter half an inch, except for one inch of its length where a cylindrical tube is inserted so as to increase the diameter of the tube to 3 inches. The bottom of the bulb is 27 inches above the mercury in the tank. The lower half of the bulb, and the tube below, contain mercury, and the upper half of the bulb, and the tube above, contain water. If the mercury barometer rises ·5 inch, through what distance will the upper surface of the water move, the sp. gr. of mercury being 13·67?

117. A balloon is filled with a gas whose sp. gr. is $\frac{1}{10}$ th that of air at a pressure of 760 mm. of mercury. Compare the lifting power of the balloon when the barometric height is 750 mm. with the lifting power when the height is 760 mm. The temperature of the air is 0°C. in both cases, and the volume of the balloon is unaltered.

118. The diameters of the pistons of a hydraulic press are 1 and 16 inches, and the velocity ratio of the lever handle is 10. What weight will the larger piston support when a force of 35 lbs. weight is applied at the end of the lever handle?

119. A rod, of section one sq. inch and of length 20 inches, is loaded at one end with lead, and floats in water with 15 inches immersed. The specific gravities of the wood and lead being ·7 and 11·3 respectively, shew that the volume of the lead is $\frac{10}{103}$ cub. in.

120. A cylindrical solid cylinder, whose external diameter is 2 inches, is loaded, and floats in water so that 12 inches of its length are below the surface. How deep will it sink in brine of specific gravity 1·05, and what weight must be added to make it sink to the same amount in the brine as it did in the water?

121. A hollow iron ball loses one-third of its weight when it is immersed in water. If the specific gravity of iron is 7·8, find what fraction of the volume of the ball is hollow.

122. A piece of wood, of volume 72 cubic inches, floats just entirely immersed in sea-water when a weight of one pound is placed on it. Given that a cubic foot of pure water weighs 62·4 pounds, and that the specific gravity of sea-water is 1·025, find the specific gravity of the wood, and the volume of the part of it that is above the water when it is floating freely and unloaded.

123. The limbs of a U-tube are each 8 inches long, and contain mercury to a height of 3 inches above the bend. Into one limb turpentine is poured until it reaches the top. Given that the specific gravities of mercury and turpentine are 13·6 and ·88 respectively, find the height of the column of turpentine.

124. Mercury (specific gravity = 13·6) is poured into the bend of a U-tube. Brine, of specific gravity 1·08, is poured into one branch above the mercury to a depth of 20 cms. What depth of water in the other branch will stand with its top level with the top of the brine?

125. A circular cylinder of metal is attached to a spring balance and lowered vertically into water. Plot a graph, shewing the change in the indication of the balance as the cylinder is gradually immersed, assuming the specific gravity of the cylinder to be 8.

126. The specific gravity of an imperfect metal casting is found to be 6·3; if the proper specific gravity of the casting is 7·5, how much per cent. of it is not occupied by metal?

127. A thin uniform rod has a very small portion of heavy metal attached to one end, and it can float in water half immersed and inclined at any angle to the horizontal; shew that the specific gravity of the rod must be $\frac{1}{4}$.

128. A rod, of small section and of density ρ , has a small portion of metal of weight $\frac{1}{n}$ th that of the rod attached to one extremity; prove that the rod will float at any inclination in a liquid of density σ if $(n+1)^2 \rho = n^2 \sigma$.

129. A cylinder, of height h and density ρ , floats with its axis vertical in a liquid of density σ_1 ; if now the density of the air increases from σ_2 to σ_3 , find by how much the cylinder will rise or sink.

130. A cylindrical piece of cork, of height h , is floating with its axis vertical in a basin of water. If the basin be placed under the receiver of an air-pump and the air be pumped out, prove that the cork will sink through a distance $\frac{\sigma}{1-\sigma}(1-s)h$, where σ and s are respectively the specific gravities of air and cork.

131. Assuming the specific gravity of air to be ·00125, and that of some brass weights to be 8·4, shew that the correction to be applied to the apparent weight of water weighed in a balance by means of these brass weights is about ·1 per cent.

132. The specific gravity corresponding to the lowest mark on the stem of a hydrometer is 1·6. What is the specific gravity corresponding to the highest mark if the reading half way between the two is 1·3?

133. An open canister, $4\frac{7}{8}$ ins. high, is inverted over a vessel of mercury and pressed down until its bottom is in the surface of the fluid. Find how high the mercury will have risen within the canister, the height of the mercury barometer being 30 ins.

134. A cylinder, of height 5 ft. with its axis vertical, is full of air at atmospheric pressure, and is closed at the top by a tightly-fitting piston of mass 30 lbs.; if the piston sinks 2 ft. under its own weight, find the thrust that must be applied to the piston to force it down an additional 2 ft.

135. A speaking tube, whose section is one square inch in area, is found to be blocked. A condensing pump is attached to the tube, and after 30 strokes the pressure of the air in the tube is found to be

4 times that of the atmosphere. If the capacity of the barrel of the pump be 50 cubic inches, shew that the obstruction is distant $41\frac{2}{3}$ feet from the mouth of the tube.

136. A cylindrical diving bell, of volume 450,000 cub. cms., is lowered into water to a depth of 1700 cms. and it is then found that an addition of 750,000 cub. cms. at atmospheric pressure is required to fill the bell. Find the height of the water barometer, and the pressure on the surface of the water inside the bell in dynes per sq. cm., the value of g being 980.

137. A cylindrical diving bell is lowered into water; neglecting the thickness of the bell, draw a curve shewing how the tension of the chain varies, starting from the position in which the bell is just immersed.

138. A piece of aluminium and a piece of lead, each weighing 200 grammes, are suspended from the pans of a balance. If the two pieces of metal are immersed in water, find the weight which must be placed in one of the pans to give equilibrium, the specific gravities of aluminium and lead being 2.65 and 11.4 respectively.

139. A body weighs 10 grammes in water and 8 grammes in a liquid of specific gravity 1.4. What is its weight in air?

140. A uniform rod, weighted at one end, floats in water with 15 cms. of its length above the surface. In alcohol it sinks 10 cms. further. Find the specific gravity of a liquid in which it floats with 10 cms. of its length above the surface, given that the specific gravity of alcohol is .88.

141. A piece of wood is weighed in air, and is then attached to a sinker, whose volume is one half that of the wood, and weighed in water. If the two weighings are the same, shew that the specific gravity of the sinker is 3.

142. A piece of an alloy of copper and aluminium weighs 100 grammes in air and 84 grammes in water. If the specific gravities of copper and aluminium are 8.8 and 2.7 respectively, find the proportions by weight of the constituents of the alloy.

143. A vertical glass tube, one metre long and sealed at the top end, is lowered into water. How far must it be lowered so that the water rises 20 cms. in the tube? [Barometric height = 760 mm. and density of mercury = 13.6.]

144. A uniform tube, 20 cms. in length, closed at the upper end, is lowered vertically into a tank of mercury until the top of the tube is level with the surface of the mercury. How far will the mercury rise inside the tube if the height of the mercury barometer is 76 cms.?

145. A barometer tube, of length 40 inches, is pushed vertically, with the open end downwards, into a vessel of mercury. If the mercury has risen to a height of 9 inches inside the tube, find the depth to which the open end has been plunged, assuming the height of the mercury barometer to be 30 inches.

146. A tank, 4 ft. 8 ins. high, is filled with water to a height of 4 feet, and an air-tight cover is then put on its top; if a small hole is made in its bottom, prove that only about one inch of the water will run out, the height of the water barometer being 34 feet.

147. A cylindrical tube of narrow bore is closed at one end and is open at the other. A quantity of air is imprisoned in it by a column of mercury 25 cms. long. When the tube is held vertically with the open end uppermost, the length of the air column is 28 cms.; when the tube is inverted, the air column increases in length to 57.6 cms.; find the barometric height.

148. Two vessels, *A* and *B*, contain equal masses of air, the volume of *A* being 20 litres and the pressure of the air in it being 15 lbs. per square inch. Communication being established between the two by opening a tap, the final pressure is found to be 20 lbs. per square inch. Find the volume of *B*.

149. A cylinder of oxygen holds 10 litres at a pressure of 20 atmospheres. What will be the pressure after enough oxygen to fill 50 litres at atmospheric pressure has been allowed to escape?

150. Mercury is poured into a long tube graduated in cub. cms. till only an air space of 5 cub. cms. is left. The tube is then closed with the finger and is inverted in a trough of mercury, and the finger removed. The imprisoned air now occupies 23 cub. cms. of the tube, and the mercury in the tube is 60 cms. above the mercury in the trough. Find the atmospheric pressure in millimetres of mercury.

151. A barometer reads 77.9 cms. when the true pressure is 78.5 cms., there being some air above the mercury occupying a length of 12 cms. What is the true pressure when the reading is 75.5 cms.?

152. A bubble of air, having a volume of 2 cubic cms. at a pressure of 75 cms. of mercury, is allowed to enter a mercury barometer, whose cross-section is 2 sq. cms. and whose vacuum is 5 cms. in length. How far will the mercury descend?

153. A body consists of a cylinder joined to a hemisphere on the same base, and floats with the spherical portion partly immersed in water; find the greatest height of the cylinder consistent with stability,

(1) if the body be solid and homogeneous,

(2) if it be hollow and made of metal which is of the same small thickness throughout.

154. A solid body consists of a right cone joined to a hemisphere on the same base, and floats with the spherical portion partly immersed; prove that the greatest height of the cone consistent with stability is $\sqrt{3}$ times the radius of the base.

155. A hollow buoy is made of a hemisphere and a cone joined at their bases, the thickness of the metal being the same throughout. Shew that it can float in stable equilibrium with the cone uppermost if the semi-vertical angle of the cone be 45° , but not if it be 30° .

TEST EXAMINATION PAPERS

A. (Chaps. I—VII)

1. State and prove the converse of the Triangle of Forces.

Apply the Polygon of Forces to shew that forces of 4 lbs. wt. acting E., 2 lbs. wt. acting S., $\sqrt{2}$ lbs. wt. acting S.W., and $3\sqrt{2}$ lbs. wt. acting N.W. are in equilibrium.

2. Find the magnitude and direction of the resultant of two forces P and Q whose directions meet at an angle a .

What is the magnitude of the resultant of two forces, equal respectively to 7 and 8 lbs. wt., which act on a particle at an angle of 60° ?

3. Find the resultant of two unequal unlike parallel forces.

A rod, 10 ft. long, whose weight may be neglected, has a mass attached to each end and balances about a point, the thrust on which is 12 lbs. wt. The mass at one end is 7 lbs. What is the other mass and where is the point?

4. Two given forces meet in a point. Prove that the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

Verify this, numerically, in the case in which the forces are represented by two of the sides of a square and the point bisects one of the other sides.

5. Define a Couple, and prove that two couples are equivalent if their moments are algebraically the same.

Prove that any number of couples in one plane are equivalent to a single couple whose moment is equal to the algebraic sum of the moments of the given couples.

6. If three forces in one plane keep a body in equilibrium, prove that they must meet in a point or be parallel.

A rod AB is horizontal and is supported by two strings, tied to it at A and B , which are inclined at 60° and 30° respectively to the vertical. Prove that the weight of the rod acts through a point C in AB , such that $AC = 3CB$. Find also the tensions of the strings in terms of the weight of the rod.

B. (Chaps. VIII—XI)

1. Define the Centre of Gravity of a body, and find its position in the case of a uniform triangular lamina.

From a thin uniform rectangle, whose sides are 6 and 8 inches respectively, a square is removed, at one corner, of side 4 inches. Find the distance of the centre of gravity of the remainder from the two sides of the rectangle that are uncut.

2. If a body be suspended from a point about which it can turn freely, prove that the centre of gravity will be vertically below the point of support.

A piece of wire, 3 ft. long, is bent into the form of 3 sides of a square and is hung up by one end. If the side attached to the point of support be inclined at an angle α to the horizon, prove that

$$\tan \alpha = \frac{4}{3}.$$

3. Find the mechanical advantage in that system of pulleys in which a separate string passes under each pulley and has one end attached to the beam from which the system is suspended; the strings are all supposed to be parallel, and the weights of the pulleys are neglected.

If there be 3 movable pulleys in the above system, and each pulley weigh 8 ozs., what effort is required to support a weight of 16 lbs.?

4. Find what horizontal force will support a body, of weight W , on a smooth plane which is inclined to the horizon at an angle α .

A weight of 7 lbs. lies on a smooth plane inclined to the horizon at an angle of 60° . A string, attached to this weight, passes over a pulley at the top of the plane. What is the greatest number of weights of 1 ounce each that can be attached to the free end of the string without making the body move up the plane?

5. Describe the Common Balance, and state what are the requisites of a good balance.

The arms of a balance are unequal in length, but its beam is horizontal when the scale-pans are empty; find the real weight of a body which, placed successively in the two scale-pans, appears to weigh 8 and 9 lbs.

6. Enunciate the Principle of Work, and prove that the work done in raising any number of material particles is the same as that done in raising a particle, equal in weight to their sum, through a distance equal to the vertical distance between the initial and final positions of their centre of gravity.

By the Principle of Work find the force required to move a truck, of weight 5 tons, up a smooth incline of 1 in 50.

C. (Chaps. XII and XIII)

1. Define Velocity and Acceleration, and prove the formulae $v = u + at$, and $s = ut + \frac{1}{2}at^2$, explaining the meaning of the symbols involved.

A certain particle, starting with a velocity of 2 feet per second and moving in a straight line, moves through 35 feet in the sixth second of its motion; determine its acceleration, assuming it to be uniform.

2. Explain carefully what is meant by the expression " $g = 32$."

A man ascends the Eiffel Tower to a certain height and drops a stone. He then ascends another 100 feet and drops another stone. The latter takes half a second longer than the former to reach the ground. Neglecting the resistance of the air, find the elevation of the man when he dropped the first stone and the time it took to drop.

3. Define the terms Mass, Gramme, and Momentum. State the three laws of Motion and give some illustration of the First Law.

What is meant by the Principle of Inertia?

On what grounds do we accept the truth of the Laws of Motion?

4. Prove the relation $P = mf$, stating carefully the meanings of the symbols, and the units in terms of which they are measured.

Define a Poundal and a Dyne, and obtain the relations between them and the weights of their corresponding units of mass.

How long would it take a poundal to stop a train whose mass is 12 tons and which has a velocity of 20 miles per hour?

5. Distinguish between Mass and Weight. Give the experiment and the reasoning by which we shew that the weights of two bodies are, at the same place, proportional to their masses.

How is it that the weight of a quantity of tea appears to be the same at all points of the earth's surface when a pair of scales is used, but that this is not the case when a spring balance is used instead?

6. Give examples of the Third Law of Motion, explaining carefully the application of the Law.

D. (Chaps. XIV—XVI)

1. Two masses, of 2 and 5 lbs. respectively, are connected by a light string hanging over a small pulley. Without using formulae find the acceleration of the system.

If the smaller mass be placed on a smooth table, and the string be laid on the table at right angles to the edge with the larger mass hanging freely, find the acceleration.

2. Describe Atwood's Machine. By its use shew how to prove that the acceleration of a given body is proportional to the force acting on it.

3. Define Impulse and Impulsive Force. If a shot be fired from a gun prove that the initial momentum of the shot is equal and opposite to that of the gun.

A bullet, of mass m , moving with horizontal velocity v , strikes, at the centre of one of its plane faces, a cubical block of wood, of mass M , which is placed on a smooth table, and remains imbedded in it.

Find the velocity with which the block commences to move.

4. Define Kinetic and Potential Energy, and give illustrations.

What is meant by the Principle of the Conservation of Energy? Prove it for the case of a particle falling freely under gravity.

5. Enunciate the proposition known as the Parallelogram of Accelerations, and deduce from it the Parallelogram of Forces.

6. Explain the principle of the Physical Independence of Forces, and give illustrations.

Apply this principle to find the velocity of projection of a ball which is thrown into the air and reaches the ground again in 3 seconds at a distance of 108 feet from the point of projection.

7. Explain from simple dynamical principles why a tricycle is very liable to be upset when it is ridden quickly round a corner of a street.

A mass of 4 lbs. revolves on a smooth table, being tied to the end of a string the other end of which is attached to the table. If the length of the string be 30 inches and the velocity of the mass 10 feet per second, what is the tension of the string in poundals?

E. (Chaps. XVII—XX)

1. Define Fluid, Liquid, Gas, and Pressure at a point.

How is it proved experimentally that pressure is transmitted equally to all parts of a fluid, and that the pressure at any point of a fluid at rest is the same in all directions?

2. Define Density and Specific Gravity, and shew how they are measured. How is the specific gravity of a mixture, consisting of known weights of fluids of known specific gravities, obtained.

The sp. grs. of two liquids are respectively 1·3 and ·8. Three lbs. wt. of the former are added to one lb. wt. of the latter. Find the sp. gr. of the resulting mixture.

3. Prove that the surface of a heavy liquid, which is at rest, is always a horizontal plane, whatever be the shape of the containing vessel.

4. Prove that the thrust on any plane surface exposed to liquid pressure is equal to the weight of a cylinder of liquid whose base is equal to the area of the given surface, and whose height is equal to the depth of the centre of gravity of the surface below the surface of the liquid.

Explain how it is that the total force exerted upon the side of any cubical cistern containing water is not proportional to the depth of the water in the cistern.

5. Prove that the thrust exerted by a fluid on any body immersed in it is equal to the weight of the fluid displaced by the body, and acts through the centre of gravity of this displaced fluid.

A piece of metal, of weight 10 lbs., floats in mercury of density 13·5 with $\frac{5}{8}$ ths of its volume immersed. Find the volume and density of the metal.

6. Define Stability of Equilibrium and Metacentre. Explain by figures the relation between the positions of the Metacentre and Centre of Gravity of a body and its Stability.

Why does an ordinary plank of wood always float in water with its length horizontal and not with its length vertical?

F. (Chaps. XXI—XXIII)

1. Explain how we obtain the sp. gr. of a liquid or solid by means of the Hydrostatic Balance.

If we wished to accurately determine the weight of some body, whose sp. gr. is very small, by using a Salter's Spring Balance, what corrections should we have to apply?

2. Shew how to find the sp. gr. of a given liquid by the use of Nicholson's Hydrometer.

A solid is placed in the upper cup of a Nicholson's Hydrometer and it is then found that 5 ozs. are required to sink the instrument to the fixed point; when the solid is placed in the lower cup 7 ozs. are required, and when the solid is taken away altogether 10 ozs. are required. What is the sp. gr. of the substance?

3. Explain Boyle's Law which connects the pressure and volume of a gas whose temperature remains constant, and shew how it can be verified in the case of the expansion of a gas.

How is it that a firmly-corked bottle full of air and immersed to a great depth in the sea will have its cork driven in?

Explain why an elastic bladder full of air would, if sunk deep enough, then sink still further if left to itself.

4. If a diving-bell be sunk into water and no additional air be supplied to it, prove that the tension of the supporting chain increases with the depth.

The height of the water barometer is 34 feet, and the depth below the surface of the water of the lowest point of a diving-bell is 68 feet. If it be now full of air, how much of this air will escape as the bell is drawn up to the surface?

5. Describe the single-barrelled Air-pump. What circumstances limit the degree of exhaustion attainable with such a pump?

6. Describe, and explain the action of, the Siphon.

What are the conditions that must hold so that it may act?

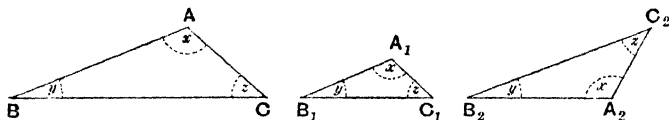
What would be the effect of piercing a small hole at the highest point of the siphon?

Why cannot a siphon be used to empty the water from the hold of a vessel which is at rest in a harbour?

APPENDIX I

Similar Triangles

1. Two triangles are said to be **equiangular** when the angles of one are respectively equal to the angles of the other.



Thus if the three triangles ABC , $A_1B_1C_1$, and $A_2B_2C_2$ have (1) the angles A , A_1 , and A_2 all equal, (2) the angles B , B_1 , and B_2 all equal, and therefore (3) the angles C , C_1 , and C_2 all equal, the triangles are equiangular.

2. The fundamental property of equiangular triangles which has been used in several articles of this book is "If two triangles are equiangular, the sides opposite the equal angles in each are proportional."

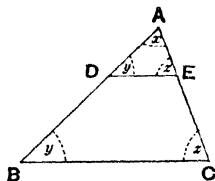
For example in the above triangles

$$\frac{AB}{A_1B_1} = \frac{BC}{B_1C_1} = \frac{CA}{C_1A_1},$$

and

$$\frac{AB}{A_2B_2} = \frac{BC}{B_2C_2} = \frac{CA}{C_2A_2}.$$

3. As a particular case of the foregoing doctrine, consider a triangle ABC in which is drawn a line DE parallel to the base.



Since DE and BC are parallel, we have

$$\angle ADE = \angle ABC.$$

So

$$\angle AED = \angle ACB.$$

The triangles ADE and ABC are therefore equiangular, so that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}.$$

Proofs of this and the preceding article will be found in any book on Geometry.

Trigonometry

4. In Geometry angles are measured in terms of a right angle. This however is an inconvenient unit of measurement on account of its size.

We therefore subdivide a right angle into 90 equal parts called **degrees**, each degree into 60 equal parts called **minutes**, and each minute into 60 equal parts called **seconds**.

The symbols used for a degree, a minute, and a second are 1° , $1'$, and $1''$.

Thus $10^\circ 11' 12''$ means an angle which is equal to 10 degrees, 11 minutes and 12 seconds.

5. Trigonometrical Ratios. Def. Let a revolving line OP start from the fixed line OA and trace out the

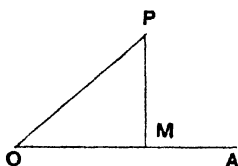


Fig. 1.

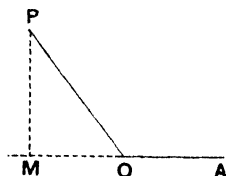


Fig. 2.

angle AOP . In the revolving line take any point P and draw PM perpendicular to the initial line OA , produced if necessary, and let it meet it in the point M .

In the triangle MOP , OP is the hypotenuse, PM is the perpendicular, and OM the base.

Then

$\frac{MP}{OP}$, i.e. $\frac{\text{Perp.}}{\text{Hyp.}}$, is called the **Sine** of the angle AOP .

$\frac{OM}{OP}$, i.e. $\frac{\text{Base}}{\text{Hyp.}}$, ... **Cosine**.....

$\frac{MP}{OM}$, i.e. $\frac{\text{Perp.}}{\text{Base}}$, ... **Tangent**

$\frac{OP}{MP}$, i.e. $\frac{\text{Hyp.}}{\text{Perp.}}$, ... **Cosecant**

$\frac{OP}{OM}$, i.e. $\frac{\text{Hyp.}}{\text{Base}}$, ... **Secant**

$\frac{OM}{MP}$, i.e. $\frac{\text{Base}}{\text{Perp.}}$, ... **Cotangent**.....

These six quantities are called the trigonometrical ratios of the angle AOP ; the three latter are not so important as the first three and have not been used in this book; we shall not refer to them any more.

6. If AOP be called θ , it is clear that

$$\sin^2 \theta + \cos^2 \theta = \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{OM^2 + MP^2}{OP^2} = 1,$$

and that $\frac{\sin \theta}{\cos \theta} = \frac{MP}{OP} \div \frac{OM}{OP} = \frac{MP}{OM} = \tan \theta$.

These two relations are very important. It follows that $\tan \theta$ is known when $\sin \theta$ and $\cos \theta$ are known.

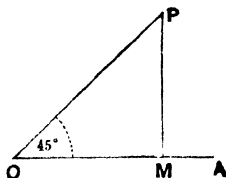
Values of the trigonometrical ratios in some useful cases

7. Angle of 45° .

Let the angle AOP traced out be 45° .

Then, since the three angles of a triangle are together equal to two right angles,

$$\begin{aligned} \angle OPM &= 180^\circ - \angle POM - \angle PMO \\ &= 180^\circ - 45^\circ - 90^\circ = 45^\circ = \angle POM. \end{aligned}$$



$$\therefore OM = MP = a \text{ (say),}$$

and

$$OP = \sqrt{OM^2 + MP^2} = \sqrt{2} \cdot a.$$

$$\therefore \sin 45^\circ = \frac{MP}{OP} = \frac{a}{\sqrt{2} \cdot a} = \frac{1}{\sqrt{2}},$$

$$\cos 45^\circ = \frac{OM}{OP} = \frac{a}{\sqrt{2} \cdot a} = \frac{1}{\sqrt{2}},$$

and

$$\tan 45^\circ = 1.$$

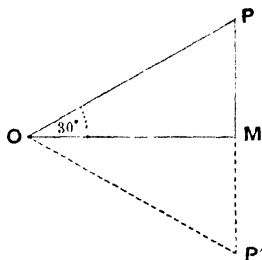
Angle of 30°.

Let the angle AOP traced out be 30° .

Produce PM to P' making MP' equal to PM .

The two triangles OMP and OMP' have their sides OM and MP equal to OM and MP' , and also the contained angles equal.

$\therefore OP = OP'$ and $\angle OPP' = \angle OP'P = 60^\circ$, so that the triangle $P'OP$ is equilateral.



Hence $OP^2 = PP'^2 = 4PM^2 = 4OP^2 - 4a^2$, where OM equals a .

$$\therefore OP = \frac{2a}{\sqrt{3}}, \text{ and } MP = \frac{1}{2} OP = \frac{a}{\sqrt{3}}.$$

$$\therefore \sin 30^\circ = \frac{MP}{OP} = \frac{1}{2},$$

$$\cos 30^\circ = \frac{OM}{OP} = a \div \frac{2a}{\sqrt{3}} = \frac{\sqrt{3}}{2},$$

and

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}.$$

Angle of 60°.

Let the angle AOP traced out be 60° .

Take a point N on OA , so that

$$MN = OM = a \text{ (say).}$$

The two triangles OMP and NMP have now the sides OM and MP equal to NM and MP respectively, and the included angles equal, so that the triangles are equal.

$$\therefore PN = OP, \text{ and}$$

$$\angle PNM = \angle POM = 60^\circ.$$

The triangle OPN is therefore equilateral, and hence

$$OP = ON = 2OM = 2a.$$

$$\therefore MP = \sqrt{OP^2 - OM^2} = \sqrt{4a^2 - a^2} = \sqrt{3} \cdot a.$$

$$\text{Hence} \quad \sin 60^\circ = \frac{MP}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2},$$

$$\text{and} \quad \tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}.$$

Angle of 0° .

Let the revolving line OP have turned through a very small angle, so that the angle MOP is very small.

The magnitude of MP is then very small and initially, before OP had turned through an angle large enough to be perceived, the quantity MP was smaller than any quantity we could assign, *i.e.* was what we denote by 0.

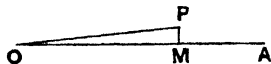
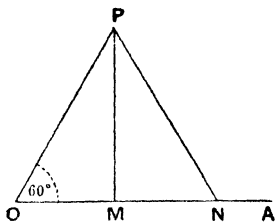
Also, in this case, the two points M and P very nearly coincide, and the smaller the angle AOP the more nearly they coincide.

Hence, when the angle AOP is actually zero, the two lengths OM and OP are equal and MP is zero.

$$\text{Hence} \quad \sin 0^\circ = \frac{MP}{OP} = \frac{0}{OP} = 0,$$

$$\cos 0^\circ = \frac{OM}{OP} = \frac{OP}{OP} = 1,$$

$$\text{and} \quad \tan 0^\circ = \frac{0}{1} = 0.$$



Angle of 90° .

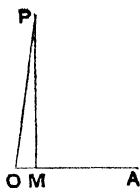
Let the angle AOP be very nearly, but not quite, a right angle.

When OP has actually described a right angle the point M coincides with O , so that then OM is zero and OP and MP are equal.

$$\text{Hence} \quad \sin 90^\circ = \frac{MP}{OP} = \frac{OP}{OP} = 1,$$

$$\cos 90^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0,$$

and $\tan 90^\circ = \frac{1}{0} = \text{what we call infinity.}$



8. To shew that $\sin (90^\circ - \theta) = \cos \theta$,
and $\cos (90^\circ - \theta) = \sin \theta$.

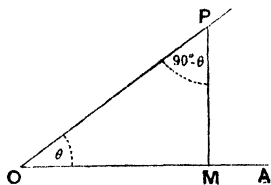
Let the angle AOP be θ .

Then $\angle OPM$

$$= 180^\circ - \angle POM - \angle OMP$$

$$= 180^\circ - \theta - 90^\circ = 90^\circ - \theta.$$

[When the angle OPM is considered, the line MO is the "perpendicular" and MP is the "base."]



$$\text{Hence} \quad \sin (90^\circ - \theta) = \sin OPM = \frac{MO}{OP} = \cos MOP = \cos \theta,$$

$$\text{and} \quad \cos (90^\circ - \theta) = \cos OPM = \frac{PM}{OP} = \sin MOP = \sin \theta.$$

Two angles, such as θ and $90^\circ - \theta$, whose sum is a right angle are said to be **complementary**.

9. In the figures of Art. 5 lines measured horizontally from O towards the right are said to be **positive**, whilst those measured from O towards the left are **negative**. Thus, in Fig. 1, OM is positive, whilst in Fig. 2 it is **negative**.

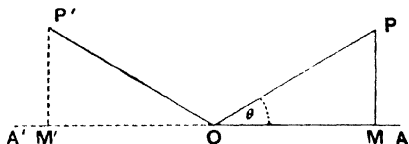
Hence, for an acute angle AOP , as in Fig. 1, the cosine is positive; for an obtuse angle, as in Fig. 2, it is negative.

Similarly, for lines in a perpendicular direction, those

drawn towards the top of the page are positive and those towards the bottom are negative.

10. To shew that $\sin(180^\circ - \theta) = \sin \theta$,
and $\cos(180^\circ - \theta) = -\cos \theta$.

We take the case only when θ is $< 90^\circ$.



Let the angle AOP be θ . Produce AO to A' and make $A'OP'$ equal to θ , so that

$$\angle AOP' = 180^\circ - \angle A'OP' = 180^\circ - \theta.$$

On OP take P' such that OP' equals OP , and draw $P'M'$ perpendicular to AOA' .

The triangles MOP and $M'OP'$ are equal in all respects, but OM' is negative whilst OM is positive.

$$\therefore M'P' = +MP,$$

and

$$OM' = -OM.$$

$$\text{Hence } \sin(180^\circ - \theta) = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin \theta,$$

and

$$\cos(180^\circ - \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos \theta.$$

So

$$\tan(180^\circ - \theta) = -\tan \theta.$$

$$\text{Exs. } \sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2},$$

$$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2},$$

$$\sin 135^\circ = \sin(180^\circ - 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}},$$

$$\cos 135^\circ = \cos(180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}},$$

and

$$\cos 150^\circ = \cos(180^\circ - 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}.$$

11. The student is advised to make himself familiar with the following table.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

If the portion of the table included between the thick lines be accurately committed to memory, the rest of it may be easily reproduced.

For, by Art. 8,

(1) the sines of 60° and 90° are respectively the cosines of 30° and 0°.

(2) the cosines of 60° and 90° are respectively the sines of 30° and 0°.

Also, by Art. 10, the sines and cosines of angles between 90° and 180° are reduced to those of angles between 0° and 90°.

Finally, the third line can be obtained from the other two since the tangent is always the sine divided by the cosine.

12. The sines and cosines of all angles between 0° and 90° are tabulated, and appear in books of mathematical Tables.

Hence, when the angle is known, its sine can be found from the Tables; so, when the sine of an angle is known, a value for the angle itself, less than a right angle, can be found. Similarly for the cosine or tangent.

13. If ABC be any triangle, the sides BC , CA , and AB which are opposite to the angles A , B , and C of the triangle are respectively denoted by a , b , and c .

In any triangle ABC , to prove that

$$b^2 = c^2 + a^2 - 2ca \cos B.$$

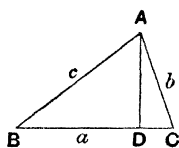


Fig. 1.

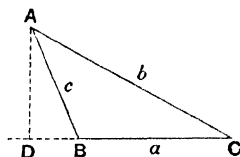


Fig. 2.

Draw AD perpendicular to BC .

Case I. Let B be acute.

By geometry, we have

$$AC^2 = CB^2 + BA^2 - 2CB \cdot BD,$$

$$\text{i.e.} \quad b^2 = a^2 + c^2 - 2a \cdot BD.$$

$$\text{But} \quad \frac{BD}{c} = \cos B, \text{ so that } BD = c \cos B.$$

$$\therefore b^2 = c^2 + a^2 - 2ca \cos B.$$

Case II. Let B be obtuse, as in Fig. 2.

By geometry, we have

$$AC^2 = CB^2 + BA^2 + 2CB \cdot BD,$$

$$\text{i.e.} \quad b^2 = a^2 + c^2 + 2a \cdot BD.$$

$$\text{But} \quad \frac{BD}{c} = \cos ABD = \cos (180^\circ - B) = -\cos B \quad (\text{Art. 10}).$$

$$\therefore b^2 = c^2 + a^2 - 2ca \cos B.$$

Whether B be acute or obtuse, we therefore have the same relation.

Similarly, we could prove that

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

$$\text{and} \quad a^2 = b^2 + c^2 - 2bc \cos A.$$

14. In any triangle to prove that the sines of the angles are proportional to the opposite sides, i.e. that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

In Fig. 1, Art. 13, we have $\frac{AD}{c} = \sin B$.

In Fig. 2, we have $\frac{AD}{c} = \sin ABD = \sin (180^\circ - B)$
 $= \sin B$ (Art. 10).

In either case, $AD = c \sin B$.

Also $\frac{AD}{b} = \sin C$, so that $AD = b \sin C$.

$$\therefore c \sin B = b \sin C.$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Similarly it can be shown that each of these is equal to

$$\frac{\sin A}{a}.$$

15. Inverse trigonometrical functions. The expression $\sin^{-1} x$ (read in words as "sine minus one x ") means the smallest angle, positive or negative, whose sine is x . Thus

$$\sin^{-1} \frac{1}{2} = 30^\circ, \text{ and } \sin^{-1} \left(-\frac{\sqrt{3}}{2} \right) = -60^\circ.$$

Similarly, $\cos^{-1} x$ means the smallest angle, lying between 0° and 180° , whose cosine is x . Thus

$$\cos^{-1} \frac{1}{\sqrt{2}} = 45^\circ, \text{ and } \cos^{-1} \left(-\frac{1}{2} \right) = 120^\circ.$$

So $\tan^{-1} x$ means the smallest angle, positive or negative, whose tangent is x . Thus

$$\tan^{-1} \sqrt{3} = 60^\circ, \text{ and } \tan^{-1} (-1) = -45^\circ.$$

APPENDIX II

TABLE OF LENGTHS, AREAS, VOLUMES, ETC.

1. The area of a Triangle = $\frac{1}{2}$ base $\times$ perpendicular on it from the opposite vertex = half the product of any two sides and the sine of the included angle.

2. The length of the circumference of a Circle of radius $r = 2\pi r$, where

$$\pi = 3.14159265\dots$$

[An approximation to the value of π is $\frac{22}{7}$, and this is the value that has been used throughout this book.]

3. The area of a Circle of radius $r = \pi r^2$.

4. The area of the surface of a Cylinder = Product of its height and the perimeter of its base.

5. Volume of a Cylinder = Product of its height and the area of its base.

6. The area of the surface of a Sphere of radius $r = 4\pi r^2$.

7. The Volume of a Sphere = $\frac{4}{3}\pi r^3$.

8. The Area of the surface of a Cone = One half the product of the perimeter of its base and its slant side = πrl .

9. The Volume of a Cone = One third the product of its height and the area of its base = $\frac{1}{3}\pi r^2 h$.

Values of "g"

<i>Place</i>	<i>Ft.-Sec. units</i>	<i>Cm.-Sec. units</i>
The Equator	32.091	978.10
London	32.191	981.17
North Pole	32.252	983.11

1 centimetre = .39370 inch = .032809 foot.

1 foot = 30.4797 centimetres.

1 litre = 1 cubic decimetre = 1000 cubic centimetres.

1 gramme = 15.432 grains = .0022046 lb.

1 lb. = 453.59 grammes.

1 poundal = 13825 dynes.

The mass of a cubic foot of water is very nearly 1000 ozs., i.e. $62\frac{1}{2}$ lbs.

The mass of a cubic centimetre of water at 4° C. is one gramme.

The sp. gr. of mercury is 13.596.

ANSWERS TO THE EXAMPLES

I. (Pages 10, 11)

1. (i) 25; (ii) $3\sqrt{3}$ ($=5.196$); (iii) 13; (iv) $\sqrt{61}$ ($=7.81$); (v) 60° ; (vi) 3.
2. 20 lbs. wt.; 4 lbs. wt.
3. $\sqrt{2}$ lbs. wt. in a direction south-west. 4. 205 lbs. wt.
5. P lbs. wt. at right angles to the first component.
6. 2 lbs. wt. 7. 60° . 8. 3 lbs. wt.; 1 lb. wt. 9. 120° .
10. In the direction of the resultant of the two given forces.
11. $-\frac{1}{4}$. 12. 12 lbs. wt.
14. (i) 23.8; (ii) 6.64; (iii) $68^\circ 12'$; (iv) 2.56.

II. (Page 13)

1. $5\sqrt{3}$ ($\Rightarrow 8.66$) and 5 lbs. wt. 2. $\frac{1}{2}P\sqrt{2}$. 3. 50 lbs. wt.
4. 16 and 12 lbs. wt. 5. $P\sqrt{3}$ and $2P$.
6. (i) 20.21 lbs. wt.; (ii) 33.62 and 51.8 lbs. wt.

III. (Pages 18, 19)

1. 1:1: $\sqrt{3}$. 2. $\sqrt{3}:1:2$. 3. 120° . 4. 40.
5. 3.46... and 1.73... lbs. wt. 6. 5:4. 7. 5 and 13.
14. $101\frac{1}{2}^\circ$; 57° . 15. 52; 95° . 16. 67.2; 101.
17. 46; 138° . 18. 29.6; 14° .
19. $R_1=34.4$ lbs. wt.; $\alpha_1=81^\circ$; $R_2=6.5$ lbs. wt.; $\alpha_2=169^\circ$.
20. Each equals 2.66 cwt.

IV. (Pages 23, 24)

1. 4 lbs. wt. in the direction AQ . 2. 9.76 lbs. wt. nearly.
3. $2P$ in the direction of the middle force.
4. $7P$. 5. $\sqrt{3}P$ at 30° with the third force.
6. $\sqrt{325+90\sqrt{3}-48\sqrt{2}-60\sqrt{6}}$ ($=16.3$) lbs. wt.
7. 14.24 lbs. wt. 8. 5 lbs. wt. opposite the second force.
9. 17.79 lbs. wt. at $66^\circ 29'$ with the fixed line.
10. 9.40 lbs. wt. at $39^\circ 45'$ with the fixed line.
11. 39.50 lbs. wt. at $111^\circ 46'$ with the fixed line.
12. 42.5 kilogs. wt. at 30° with OA . 13. 42.9 and 19.91 lbs. wt.

V. (Pages 26, 27)

1. $\frac{W}{2}(\sqrt{6}-\sqrt{2}) (=518W)$; $W(\sqrt{3}-1) (=732W)$.
2. $2\frac{2}{3}$ and $3\frac{1}{3}$ lbs. wt.
3. 126 and 32 lbs. wt.
4. 56 and 42 lbs. wt.
5. 48 and 36 lbs. wt.
6. 4, 8, and 12 lbs. wt.
7. W .
8. 120 lbs. wt.
9. 1.34 ins.
10. 14 lbs. wt.
11. 6 ft. 5 ins.; 2 ft. 4 ins.
12. They are each equal to the weight of the body.

VI. (Pages 33, 34)

1. (i) $R=11$, $AC=7$ ins.; (ii) $R=30$, $AC=1$ ft. 7 ins.;
(iii) $R=10$, $AC=1$ ft. 6 ins.
2. (i) $R=8$, $AC=25$ ins.; (ii) $R=8$, $AC=-75$ ins.;
(iii) $R=17$, $AC=-19\frac{1}{17}$ ins.
3. (i) $Q=9$, $AB=8\frac{1}{2}$ ins.; (ii) $P=2\frac{3}{4}$, $R=13\frac{3}{4}$;
(iii) $Q=6\frac{3}{4}$, $R=12\frac{3}{4}$.
4. (i) $Q=25$, $AB=3\frac{3}{50}$ ins.; (ii) $P=24\frac{3}{4}$, $R=13\frac{3}{4}$;
(iii) $Q=2\frac{1}{4}$, $R=3\frac{3}{7}$.
5. 15 and 5 lbs. wt.
6. $43\frac{1}{3}$ and $13\frac{1}{3}$ lbs. wt.
7. 98 and 70 lbs. wt.
8. The block must be 2 ft. from the stronger man.
9. 4 ft. 3 ins.
10. 1 lb. wt.
11. 1 foot.
12. 20 lbs.; 4 ins.; 8 ins.
13. $14\frac{3}{8}$ ins.; $10\frac{5}{8}$ ins.
15. $\frac{1}{2}W$.
16. (1) 100 and 150 lbs. wt.; (2) 50 and 100 lbs. wt.; (3) 25 and 75 lbs. wt.
17. 77.55 and 34.45 lbs. approximately.

VII. (Pages 42, 43)

1. 10.1.
2. $75\sqrt{3} (=129.9)$ lbs. wt.
3. 3 ft. 8 ins. from the 6 lb. wt.
4. At a point distant 6.6 feet from the 20 lbs. wt.
5. $2\frac{9}{7}$ ft. from the end.
6. $2\frac{2}{3}$ lbs.
7. $2\frac{1}{3}$ lbs.
8. (1) 4 tons wt. each; (2) $4\frac{1}{3}$ tons wt., $3\frac{2}{3}$ tons wt.
9. B is 3 inches from the nearest peg.
10. $\frac{6}{7}$ cwt.
11. One-quarter of the length of the beam.
12. The weight is $3\frac{1}{2}$ lbs. and the point is $8\frac{1}{2}$ ins. from the 5 lb. wt.
13. 3 ozs.
14. $85\frac{1}{2}$, $85\frac{1}{2}$, and 29 lbs. wt.
15. 96, 96, and 46 lbs. wt.
16. $11\frac{7}{8}$ ins. from the axle.
19. $3\frac{1}{8}$, 2, and $1\frac{1}{8}$ tons wt.

VIII. (Page 47)

2. 9 ft.-lbs.
3. 6.
4. A force equal, parallel, and opposite, to the force at C , and acting at a point C' in AC , such that CC' is $\frac{2}{3}AB$.

IX. (Pages 50—52)

2. 45° .
3. $10\sqrt{2}$ and 10 lbs. wt.
4. The length of the string is AC .
5. $\frac{2}{3}W\sqrt{3}$; $\frac{1}{3}W\sqrt{3}$.
7. $46\frac{2}{3}$ and $10\frac{1}{3}\sqrt{421}$ ($=68.4$) lbs. wt.
9. $W \operatorname{cosec} \alpha$ and $W \operatorname{cota}$.
10. $\frac{1}{3}W\sqrt{3}$, where W is the weight of the sphere.
11. 30° ; $\frac{2}{3}W\sqrt{3}$; $\frac{1}{3}W\sqrt{3}$.
12. $\sqrt{21} : 6$.
13. $4\frac{1}{3}\sqrt{3}$ ($=23.094$) lbs. wt.
14. $6\frac{2}{3}$ lbs. wt.
16. 3.16 ft.; 133 and 118.8 lbs. wt.
17. 15.5 lbs. wt.
18. 6.75 and 16.6 lbs. wt.
19. 2.83 and 3.61 cwt.
20. 26.8 and 32.1 lbs. wt.

X. (Page 57)

1. $2\frac{2}{3}$ ft.; $\frac{1}{3}\sqrt{97}$ ($=3.283\dots$) ft.
2. $3\frac{1}{3}$ ft.; $\frac{2}{3}\sqrt{73}$ ($=5.696$) ft.; $\frac{4}{3}\sqrt{13}$ ($=4.807$) ft.
6. 60° .

XI. (Pages 59, 60)

1. $4\frac{2}{3}$ inches from the end.
2. 15 inches from the end.
3. $2\frac{5}{9}$ feet.
4. $2\frac{1}{2}$ inch from the middle.
5. $7\frac{1}{3}$ ins. from the first particle.
6. $4\frac{2}{5}$ ins. from the centre of the florin.
7. $5 : 1$.
8. 3.5 ins. from the base.
9. 12 lbs.; the middle point of the rod.

XII. (Pages 63, 64)

1. One-fifth of the side of the square.
2. It is distant $\frac{3a}{4}$ from AB and $\frac{a}{4}$ from AD , where a is the side of the square.
3. At a point whose distances from AB and AD are respectively 16 and 15 inches.
4. $7\frac{1}{3}$ and $8\frac{1}{3}$ inches.
5. $\frac{a}{6}\sqrt{19}$ ($=.73a$); $\frac{a}{30}\sqrt{283}$ ($=.56a$).
7. At the centre of gravity of the lamina.
8. $8\frac{1}{3}$ and $11\frac{1}{3}$ inches.
11. $4\frac{1}{3}$ inches from A .
12. One-quarter of the side of the square.

XIII. (Pages 66, 67)

1. $2\frac{1}{24}$ inches from the joint.
2. $5\frac{7}{8}$ inches from the lower end of the figure.
3. It divides the beam in the ratio of $5 : 11$.
4. At the centre of the base of the triangle.
5. $7\frac{3}{8}$ inches.

6. Its distance from the centre of the parallelogram is one-ninth of a side.

7. The distance from the centre is one-twelfth of a diagonal.

8. The distance from the centre is $\frac{1}{2\sqrt{3}}$ th of the diagonal.

9. It divides the line joining the middle points of the opposite parallel sides in the ratio of 5 : 7.

10. $1\frac{7}{10}$ inches.

11. $\frac{15}{194}$ inch from the centre.

12. The centre of the hole must be 16 inches from the centre of the disc.

13. It is one inch from the centre of the larger sphere.

14. 13.532 inches.

15. 3080 miles nearly.

XIV. (Pages 73, 74)

1. $6\frac{2}{3}$ inches.

2. $\frac{1}{10}\sqrt{10}$ feet (= 3.8 inches) nearly.

3. 15a.

4. The weight of the table.

5. On the line joining the centre to the leg which is opposite to the missing leg, and at a distance from the centre equal to one-third of the diagonal of the square.

6. 120 lbs.

7. $\frac{W}{6}$.

10. 18 if the bricks overlap in the direction of their lengths, and 8 if in the direction of their breadths.

12. $\frac{5}{8}$ ths of the diameter of a penny.

XV. (Pages 78, 79)

1. 5 feet.

2. 4 feet from the first weight; toward the first

weight.

3. 11 : 9.

4. 2 lbs.

6. 6 ins. from the 27 ounces; $1\frac{5}{7}$ inches.

7. 360 stone wt.

8. 50 lbs. wt.

10. 20 lbs. wt.

11. A force equal to the weight of $2\frac{1}{2}$ cwt.

XVI. (Pages 83, 84)

1. (1) 320; (2) 7; (3) 3.

2. (1) 7; (2) $45\frac{1}{2}$; (3) 7; (4) 6.

3. 290 lbs.

4. $10\frac{3}{4}$ lbs.

5. 5 lbs.

6. 5 lbs.

7. 4w; 21w.

8. $9\frac{15}{17}$ lbs. wt.

9. 18 lbs. wt.

XVII. (Page 85)

1. 6 lbs.

2. 4 strings; 2 lbs.

3. 47 lbs.; 6 pulleys.

4. 7 strings; 14 lbs.

5. $\frac{W}{n+1}$, where n is the number of strings.

6. 9 stone wt.

7. The cable would support $2\frac{1}{4}$ tons.

8. 75 lbs.; $166\frac{2}{3}$ lbs.

XVIII. (Pages 88, 89)

1. (1) 30 lbs.; (2) 4 lbs.; (3) 4.
2. (1) 161 lbs.; (2) 16 lbs. wt.; (3) $\frac{1}{2}$ lb.; (4) 5.
3. 10 lbs. wt.; the point required divides the distance between the first two strings in the ratio of 23 : 5.
4. $\frac{11}{16}$ inch from the end. 5. $18\frac{6}{53}$. 6. $\frac{4}{7}$ inch from the end.
7. $W = 7P + 4w$; 8 ozs.; 1 lb. wt. 8. 4; 1050 lbs.

XIX. (Pages 91, 92)

1. 12 lbs. wt.; 20 lbs. wt. 2. 30° ; $W\frac{\sqrt{3}}{2}$.
3. 103.92 lbs. wt. 5. $\sqrt{3} : 1$. 6. 3 : 4; $2P$.
7. $\frac{\sqrt{3}}{3}$, i.e. .577 lb. wt.; $\frac{7\sqrt{3}}{3}$, i.e. 4.041 lbs. wt. 8. 6 lbs. wt.
9. $\frac{\sin \alpha}{\sin \beta - \sin \alpha}$ tons. 10. 17.374 lbs. wt.; 46.884 lbs. wt.
11. 10.318 lbs. wt.; 12.208 lbs. wt. 12. 16.12 lbs. wt.; 34.056 lbs. wt.

XX. (Pages 94, 95)

1. 7 lbs. wt. 2. 120 lbs. wt.; 140 lbs. wt.; $110\frac{10}{11}$ lbs. wt.
3. 20 inches. 4. 7 feet. 5. $3\frac{4}{5}$ tons. 6. 3 lbs. wt.

XXI. (Page 98)

1. 11 lbs. 2. $26\frac{1}{4}$ lbs. 3. 2 ozs.
4. 2 : 3; 6 lbs. 5. 24.494 lbs. 6. $5 : \sqrt{26}$.
7. $\frac{6}{5}\sqrt{110}$ (= 12.59) inches; $\sqrt{110}$ (= 10.49) lbs.
9. 2s. 3d.; 1s. $9\frac{1}{3}$ d. 10. He will lose one shilling.

XXII. (Page 102)

1. $34\frac{5}{6}$ inches from the fulcrum.
2. 2 inches from the end; 1 inch. 3. 32 inches from the fulcrum.
4. 4 inches. 5. 26 lbs.; 14 lbs.; 10 ins. from the fulcrum.
6. 3 ozs. 7. 30 inches.
9. The machine being graduated to shew lbs., the weights indicated must each be increased by $\frac{1}{6}$ th of a lb.
10. He cheats his customers, or himself, according as he decreases, or increases, the movable weight.

XXIII. (Pages 109, 110)

1. 10 lbs. wt.; $10\sqrt{17}$, i.e. 41.231 lbs. wt. at an angle, whose tangent is 4, i.e. $75^\circ 58'$, with the horizontal.
2. $\frac{P}{W} = \frac{\sqrt{2}}{3} = .4714$.

3. $10\sqrt{10}$, i.e. 31.623 lbs. wt. at an angle, whose tangent is 3, i.e. $71^\circ 34'$, with the horizon.

5. $\frac{1}{4}$. 6. $\frac{4}{3}\sqrt{3}$ lbs. wt. 8. $\frac{\sqrt{3}}{15}$. 9. 2.19 cwt.
 10. 79.7 lbs. wt.; .32. 12. $a \times .134$. 13. $W \frac{1+\mu}{1-\mu}$; $W \frac{1-\mu}{1+\mu}$; $\frac{1}{3}$.
 15. At an angle equal to the angle of friction.

XXIV. (Page 118)

1. 4400 lbs. 2. $5\frac{8}{11}$ inches. 3. $\frac{25}{27}$ lbs. wt.
 4. $1\frac{69}{76}$ lbs. wt. 5. $4\frac{94}{99}$ lbs. wt. 6. $13\frac{23}{49}$ tons wt.

XXV. (Pages 119, 120)

1. 21120. 2. 23,040,000 ft.-lbs.; $5\frac{9}{11}$ H.P.
 3. 1000 feet. 4. $9\frac{27}{32}$ hours. 5. $8\frac{13}{14}$.
 6. 4.4352. 7. 660,000 ft.-lbs.; 30 H.P.
 8. 1500 ft.-lbs. 10. $111\frac{17}{28}$ tons wt.
 12. 176 ft.-lbs.; .213 H.P. 13. 3808 ft.-lbs.
 14. 7,392,000 ft.-lbs.; $7\frac{7}{15}$ H.P.

XXVI. (Pages 127, 128)

1. (1) 17 ft. per sec.; $47\frac{1}{2}$ feet. (2) 0; $24\frac{1}{2}$ feet.
 (3) $-\frac{55}{18}$; $1\frac{7}{11}$ secs. (4) 3 ft. per sec.; 6 secs.
 2. 40 ft. per sec.; 400 ft. 3. 40 secs. 4. 20 ft.-sec. units.
 5. 10 secs.; 150 cms. 6. In 50 secs.; 25 metres.
 7. 18 ft.-sec. units. 8. 10 ft. per sec.; $-\frac{1}{5}$ ft.-sec. unit.
 9. 19 ft. per sec.; 3 ft.-sec. units; $60\frac{1}{8}$ ft. 10. 5 secs.; $12\frac{1}{2}$ ft.
 11. 16 ft.-sec. units; 30 ft. per sec.
 12. 30 ft. per sec.; -2 ft.-sec. units. 13. 30 ft.
 14. $\frac{1}{3}$, $\frac{\sqrt{2}-1}{3}$ ($=.138$), and $\frac{\sqrt{3}-\sqrt{2}}{3}$ ($=.106$) secs. respectively.
 15. In 2 secs. at 16 ft. from O. 16. Yes. 17. 10 secs. or 30 secs.

XXVII. (Pages 131-133)

1. 25 ft.; $\frac{1}{4}$ sec. and $2\frac{1}{4}$ secs. 2. (1) In $\frac{5}{8}$ secs.; (2) in $1\frac{1}{4}$ secs.
 3. In $1\frac{1}{4}$ and $2\frac{1}{2}$ secs.; 50 ft.
 4. (1) 1600 ft.; (2) $\frac{1}{4}\sqrt{10}$ ($=.79$) sec.; (3) 60 ft. per sec. upwards.
 5. 432 ft. 6. 44 secs. 7. 2 secs. or $5\frac{1}{2}$ secs.
 8. 545 cms. per sec.; $\frac{4}{9}$ sec. 9. 10.2 secs.
 10. 218 metres; $6\frac{2}{3}$ secs. 11. 32.18. 12. 900 ft.; $7\frac{1}{2}$ secs.
 13. 100 ft. 14. 150 ft. 15. 144 ft.
 16. 256 ft. per sec.; 1024 ft. 17. $t=5$; 64 ft. per sec.
 18. 784 ft. 19. 1120 ft. per sec. 20. 4080 ft.

21. $68\frac{4}{3}$ ft. per sec.; $306\frac{18}{23}$ ft. 22. 1 sec.; $1\frac{1}{2}$ secs.
23. 576 ft.; 2 secs. 24. $\cdot 77\dots$ sec.; 217:162.

XXVIII. (Pages 139—141)

1. (1) $\frac{1}{2}$, (2) $\frac{9}{2}$, (3) $\frac{9}{448}$ ft.-sec. units.
2. (1) 200 poundals; (2) $6\frac{1}{4}$ lbs. wt. 3. 15 lbs. wt.
4. $15\frac{2}{3}$ lbs. weight. 5. 48 ft.-sec. units; 720 feet.
6. 1:64; 5 ft. per sec. 7. $7\frac{1}{2}$ secs.; $13\frac{1}{3}$ ft. per sec.
8. 2 mins. 56 secs. 9. 14 secs. 11. 180 feet.
12. $\frac{1}{2}$ tons wt.; $\frac{1}{2}$ tons wt.
13. $363\frac{1}{3}$ cms. per sec.; $181\frac{2}{3}$ cms.; 21,800 cms.
14. 49.05 kilogrammes. 15. 55:2. 16. 144 lbs.
17. 12 lbs. 18. $7\frac{1}{16}$ lbs. wt., $237\frac{3}{8}$ lbs. wt.
19. They are equal. 20. 110 lbs. wt. 21. $133\frac{1}{3}$ ft. per sec.

XXIX. (Pages 147, 148)

1. $\frac{9}{5}$; $7\frac{1}{5}$ lbs. wt.
2. (1) 4 ft.-sec. units; (2) $7\frac{7}{8}$ lbs. wt.; (3) 20 ft. per sec.; (4) 50 ft.
3. 18 ft.; $15\frac{3}{4}$ ozs. wt. 4. $x=985$. 6. By 2 lbs. wt.
7. $\frac{m}{2}$. 8. 16 ft.
9. (1) $\frac{9}{10}$; (2) $\sqrt{5}$ ($=2\cdot 236$) secs.; (3) $\frac{16}{5}\sqrt{5}$ ($=7\cdot 155$) ft. per sec.
10. $\frac{5g}{24}$; $3\frac{23}{24}$ ozs. wt. 11. 2 secs. 12. 125 grammes.
13. In the ratio 19:13. 14. $2\frac{1}{2}$ and $3\frac{1}{3}$ lbs. wt.; $\frac{g}{6}$.
15. 29 ft. 9 ins. nearly.

XXX. (Pages 151—153)

1. 200 ft.; 5 secs.
2. $16\sqrt{3}$ ($=27\cdot 71$) ft. per sec.; $\frac{5}{8}\sqrt{3}$ ($=1\cdot 08$) secs. 3. 30° .
4. 1:4. 5. $\sqrt{30}$ ($=5\cdot 477$) secs.; $16\sqrt{30}$ ($=87\cdot 635$) ft. per sec.
6. 30° .
7. (1) 2 ft.-sec. units; (2) $21\frac{3}{16}$ lbs. wt.; (3) 6 ft. per sec.; (4) 9 ft.
8. 40 ft. 9. 24 lbs. 10 ozs. 10. 605:18.
11. (1) 5 mins. 8 secs.; (2) 6776 feet.
12. 1 min. $42\frac{2}{3}$ secs.; $2258\frac{2}{3}$ feet. 13. $5\frac{20}{27}$ tons wt.
14. 1 mile 1408 yds. 15. $1224\frac{2}{3}$ yds.; $411\frac{6}{125}$ ft. 16. $\frac{1}{26}g$; $\cdot 3$.
17. $\cdot 1$. 18. $\frac{\sqrt{2}}{2}$ ($=\cdot 71$) sec.; $8\sqrt{2}$ ($=11\cdot 31$) ft. per sec.
19. $\frac{1}{2}\sqrt{5}$ ($=1\cdot 12$) secs.; $\frac{16}{5}\sqrt{5}$ ($=7\cdot 16$) ft. per sec.

XXXI. (Page 157)

1. Nothing.
2. (1) 20 lbs. wt.; (2) $20\frac{5}{8}$ lbs. wt.
3. (1) 154 lbs. wt.; (2) 70 lbs. wt.
4. $\frac{9}{8}$.
5. $2057\frac{1}{7}$ feet.
6. $\frac{1}{2}$ lb. wt.; $\frac{5}{12}$ lb. wt.; $\frac{1}{3}$ lb. wt.
7. $3\frac{3}{4}$ ozs. wt.; $\frac{9}{4}$; $2\frac{1}{2}$ ozs. wt.; 3 ozs. wt.

XXXII. (Pages 161, 162)

1. $4\frac{2}{7}$ ft. per sec.
4. $6\frac{1}{4}$ ft. per sec.
5. $17\frac{6}{7}$ ft. per sec.
6. 6.8... ft.
7. 1431 ft. per sec. nearly.
8. wt. of 104 cwt.
9. The masses move with a velocity of 24 ft. per sec.
11. 39.4 lbs. wt. nearly.
12. 7500 lbs. wt.

XXXIII. (Page 164)

1. 160.
2. $213\frac{1}{3}$.
3. 119.46.
4. 14.685 lbs. wt.
5. $21\frac{3}{5}$.
6. $68\frac{6}{135}$.
7. 152 ft.-lbs.

XXXIV. (Page 168)

1. (1) 5120, (2) 1280, (3) 0, units of energy.
2. 15,625.
3. 125×10^8 .
4. $37\frac{2}{3}$.
5. 625×10^{10} ; 3125×10^8 .
6. $31604\frac{0}{1}$ ft.-lbs.
7. (1) They are equal; (2) they are in the ratio $m : M$.
8. 83.36.
9. 1.91.
10. 1 : 2; 15 : 1; 300 and 600 poundals; 600 and 20 feet.

XXXV. (Pages 173, 174)

2. 100 ft.
3. 120° .
5. At an angle whose cosine is $-\frac{3}{5}$, i.e. $126^\circ 52'$, with the current; perpendicular to the current so that his resultant direction makes an angle whose tangent is $\frac{5}{3}$, i.e. $59^\circ 2'$, with the current.
6. $4\sqrt{3}$ miles per hour; 12 miles per hour.
7. $\sqrt{29}$ at an angle of elevation, whose tangent is $\frac{2}{5}$, i.e. $21^\circ 48'$, above a horizontal line which is inclined at an angle, whose tangent is $\frac{4}{3}$, i.e. $53^\circ 7'$, north of east.
8. 60° .
9. 14 at an angle whose cosine is $\frac{1}{4}$, i.e. $21^\circ 27'$, with the greatest velocity.

XXXVI. (Page 176)

1. 55 ft. per sec. at an angle $180^\circ - \tan^{-1} \frac{3}{4}$ ($=143^\circ 8'$) with the direction of the train's motion.
2. 20 miles per hour at an angle $\tan^{-1} \frac{3}{4}$ ($=36^\circ 52'$) west of north.
3. 15 miles per hour north-east.

4. 39 miles per hour in a direction $\cos^{-1} \frac{5}{13} (= 67^\circ 23')$ north of east.
5. $32\frac{8}{11}$ miles per hour.
6. $7\sqrt{5} - 2\sqrt{2} (= 10.32)$ miles per hour. Draw $OA (= 14)$ towards the east and $OB (= 7)$ towards the south-east and complete the parallelogram $OABC$. Then OC is the required direction.
7. $5\frac{5}{11}$ secs. 8. 10 miles per hour south-east.
9. 10.7 miles per hour at $7^\circ 29'$ east of south.

XXXVII. (Pages 183, 184)

1. (1) 16 ft.; 2 secs.; 110.9 ft. (2) 75 ft.; 4.33... secs.; 173.2 ft.
2. $1333\frac{1}{3}$ ft. per sec.
3. 50.1 at an angle, whose tangent is $\frac{6}{11}$, i.e. $28^\circ 37'$, to the horizon.
4. (1) $16\sqrt{17} (= 65.97)$ ft. per sec. at an angle, whose tangent is 4, i.e. $75^\circ 58'$, to the horizon.
- (2) $16\sqrt{37} (= 97.32)$ ft. per sec. at an angle, whose tangent is 6, i.e. $80^\circ 32'$, to the horizon.
6. $2\frac{1}{2}$ secs.; $146\frac{2}{3}$ ft. 7. $2h$; $2\sqrt{gh}$. 8. 5543 yards nearly.
9. 13 secs.; 3328 ft. 10. A circle of about 91 miles radius.

XXXVIII. (Page 186)

1. $14\frac{1}{16}$ lbs. wt. 2. 25 lbs. wt.
3. $\sqrt{4905}$, i.e. about 70, cms. per sec. 4. 16 ft. per sec.
5. $628\frac{5}{6}$ lbs. wt. 6. $2\frac{2}{5}\frac{1}{10}$ tons wt.

XXXIX. (Page 195)

1. 156.25 kilos. 2. 5.6 lbs. wt. 3. $2\frac{2}{4}\frac{2}{8}$ lbs. wt.
4. 7 : 1. 5. $3\frac{13}{54}$ lbs. wt.; $\frac{2439}{7}\pi = 1091\frac{1}{9}$ tons wt.
7. 144 lbs. wt. per sq. inch.
8. $\frac{128}{\pi} = 40\frac{8}{11}$ lbs. wt. per sq. inch. 9. 80 lbs. wt. per sq. inch.

XL. (Page 199)

1. $562\frac{1}{2}$ lbs. wt. 2. 4.629.... 3. 135.98 lbs. wt.
4. 13,600 grammes wt. 5. 2.6. 6. $1\frac{237}{275}$ cub. ft.
7. Its volume is increased by 1.153... cub. cms.
8. 5.72..., taking $\pi = \frac{22}{7}$. 9. $\frac{9}{275}$ sq. in.
10. .015625 cub. metre. 11. 8.721.... 12. 49.9....

XLI. (Page 202)

1. In ratio 1 : 3. 2. $\frac{6}{7}$ cub. ft. 3. 15 ozs.
4. $\frac{1}{4}(\rho_1 + \rho_2 + 2\rho_3)$. 5. 6 and 2. 6. .9375.
7. $\frac{249}{1815}$ cub. cm. 9. 1.067.

XLII. (Pages 208, 209)

1. $2291\frac{2}{3}$ lbs. wt.
2. $195\cdot84$ ft.
3. $7\frac{3}{4}$ ft.
4. $36\cdot864$ ft.
5. 4 miles $1561\cdot6$ yds.
6. $2833\frac{1}{8}$ lbs. wt.
7. 98 ft.
8. 54 ft.
9. $14\frac{109}{144}$.
10. $15\frac{5}{8}$.
11. $2021\cdot04$ grains wt.
12. $13\frac{67}{432}$.
13. $14\cdot9556$ cub. ins.

XLIII. (Pages 211—213)

1. 750 lbs. wt.
2. $162\frac{73}{90}$ lbs. wt.
3. $156\frac{1}{4}$ lbs. wt. on the upper face; $218\frac{3}{4}$ lbs. wt. on the lower face; $187\frac{1}{2}$ lbs. wt. on each vertical face.
4. 320 lbs. wt.
5. $255\frac{1}{2}$ cwt.
6. $187\frac{1}{2}$ lbs. wt.
8. $104\frac{13}{28}$ tons wt.
9. $112\frac{3}{32}$ lbs. wt.
10. $15,066\frac{27}{32}$ tons wt.
11. $\frac{125}{288}$ lbs. wt. per sq. in.; $\frac{125}{18}\pi (=21\frac{52}{3})$ lbs. wt.
12. It divides the vertical sides in the ratio $\sqrt{2}+1:1$.
13. 1·2 kilogs. wt.
14. $515\frac{5}{8}$ lbs.
15. $\pi \times \frac{125\sqrt{26}}{416} (=4\cdot81)$ lbs. wt.; $\pi \times \frac{6375\sqrt{26}}{416} (=245\cdot46)$ lbs. wt.
16. $\frac{125}{16}\pi (=24\frac{31}{8})$ lbs. wt.; $\frac{625}{8}\pi (=245\frac{15}{8})$ lbs. wt.
17. $6\frac{2}{5}$ ft.; $1\frac{3}{5}$ ft.
18. 1250 and $1312\frac{1}{2}$ lbs. wt. respectively.

XLIV. (Pages 220—222)

1. $2\frac{6073}{10800}$ cub. ft.
2. $10\frac{93}{224}$ ozs., taking $\pi = \frac{22}{7}$.
3. $\frac{25}{62}$ cub. ft.
4. 50 cub. cms.
5. $\frac{4}{5}$; $\cdot0005\bar{3}$.
6. $45\frac{5}{1111}$ cub. metres.
7. 6608·4 cub. ft.
8. $\frac{7}{25}$.
9. $31\frac{1171}{1559}$ cub. cms.; 8·66i.
10. $257\frac{41}{107}$ ft.
11. $\cdot726\dots$ inch.
13. $4\frac{56}{61}$ ins.
15. $\cdot25$.
16. They are equal.
17. $4\frac{3}{4}$ ins.
18. There is a cavity of volume 1 cub. cm.
19. $463\frac{1}{209}$ cub. ins.
20. The edge of the cube is 28·8 ins.
21. $1\frac{1}{3}$.
22. 30 lbs.
23. 900 cub. ins.; 10 ins.

XLV. (Pages 223, 224)

1. $\cdot50065$.
2. $\frac{2}{3}$ cub. in.
3. 18·9 cub. inches.
4. $13\cdot6054\dots$
5. One half.
6. It will sink.
7. It will rise.
8. It will be lessened.
10. The new depth of immersion is to the original depth as 3935:3948.

XLVI. (Pages 225, 226)

1. (1) 12 lbs. wt.; (2) 6 lbs. wt. 4. 155 : 187.
5. 37380 : 37249. 6. $97\frac{5}{19}$ lbs. wt.; $145\frac{1}{9}$ lbs. wt.
7. $\frac{69}{146}$ oz. 8. 18·5. 9. 57 grains. 10. 27 : 23.
11. The piece of wood. 12. 5.
13. 2 cub. ins.; $\frac{875}{1728}$ lbs. wt. 14. $7\frac{1}{2}$ lbs. wt.; 56 lbs. wt.

XLVII. (Pages 227, 228)

1. 3 ozs. wt. 2. 2 lbs. 6 ozs. 5. 5 lbs. wt. 6. 1,580,000 grammes.
7. $114\frac{7}{8}$ ozs. wt. 8. $\frac{3}{28}$ g. 9. 161·17 tons.

XLVIII. (Page 234)

1. ·75. 2. ·7864 nearly. 3. $7\frac{9}{13}$. 4. 2·0458... 5. $6\frac{1}{4}$.

XLIX. (Pages 237, 238)

1. 1·525. 2. 3. 3. $2\frac{6}{19}$. 4. ·865. 5. $1\frac{2}{3}$. 6. $\frac{2}{3}$.
7. $\frac{5}{6}$. 8. ·848. 9. ·9413 nearly. 10. 1·841.
11. ·87; 50 cub. cms. 12. 30 grms. wt.; 2. 13. ·5.
14. $1\frac{1}{2}$. 15. $7\frac{7}{64}$ lbs.; $5\frac{1}{17}$ lbs. of gold and $1\frac{2}{34}$ lbs. of silver.

L. (Page 242)

1. 3·456, 3·1418 and 2·88. 2. 1·03. 3. $\frac{1}{501} \cdot \frac{s}{s-1}$.
4. 10 : 13. 5. 18 : 19. 6. $8\frac{7}{20}$ ozs. 7. 2·5.
8. 8. 9. $2\frac{2}{7}$ ozs.

LI. (Page 243)

1. 27·2. 2. 6 inches.
3. At the bottom of the vertical tube containing the oil

LII. (Page 249)

1. 1169·256 cms. 2. 929,082 grms. wt., taking $\pi = 2\frac{2}{7}$
3. $17\frac{127}{1792}$ tons wt.
4. $1\frac{4}{3}$; the height would be lessened by a distance x , such that the weight of the mercury in a length x of the tube would equal the weight of the bullet. This assumes that the bullet fits the tube. If it floats in the mercury there would be no alteration in the height.
5. 2·623... cms. 6. $1\frac{1}{81}$ inches.

LIII. (Pages 255—257)

1. ·001292. 2. An increase of $8\frac{4}{9}$ grains wt.
3. 25·92 lbs. 4. 31·5 feet. 5. ·00007764... cub. in.
6. Till the level of the water inside is 68 feet below the surface of the water.

7. 32.75 ft. 8. 63 cms.
 10. The pressures on the two faces are $56\frac{1}{4}$ and $22\frac{1}{2}$ lbs. wt. per square inch; 8 inches.
 11. (1) It would float; (2) it would sink. 13. $\frac{7}{8}$ cub. in.
 14. 5 inches. 15. 29.98 inches. 16. $32\frac{1}{4}$ inches.
 18. The pressure is that due to $63\frac{1}{2}$ inches of mercury; $10\frac{4}{5}\frac{5}{9}$ ins.
 19. 34.4 lbs. wt. nearly. 22. 7.5; 30 ft.

LIV. (Page 260)

1. $8\frac{1}{2}$ cub. inches. 2. 10 cub. inches. 3. 429 : 224. 4. 15° C.

LV. (Pages 263—265)

1. 3.45... atmospheres, nearly. 2. $1\frac{1}{8}$ ft. 3. $14\frac{1}{8}$ ft.
 4. $1097\frac{309}{1003}$. 5. 20 ft.; $132\frac{6}{17}$ cub. ft. 6. 500 cub. ft.
 7. The quantities are as 3 : 2.
 8. The depth of the top of the bell is 3 inches; the height of the water-barometer is 33 ft.
 9. $33\frac{1}{3}$ ins.; 3 ft. 9 ins. 10. It remains constant.
 14. The air will flow out.
 16. (1) Some air will flow out; (2) there will be equilibrium;
 (3) some water will flow in.

LVI. (Page 270)

1. The height varies from 31.73 to 35.13 feet.
 2. 42 ft. 1 in. 3. 33 ft. 4 ins. 4. 80. 5. It will.
 6. 2 feet; $32 - 16\sqrt{2} (= 9.37)$ feet nearly. 7. $8680\frac{5}{9}$ lbs. wt.
 8. $260\frac{5}{12}$ lbs. wt.
 9. $\frac{625\pi}{8} (= 245.54)$ lbs. wt.; $\frac{3125\pi}{8} (= 1227.7)$ lbs. wt.

LVII. (Pages 275, 276)

1. 3 : 1. 2. They are as $9^8 : 10^8$.
 4. The final pressure is to the original pressure as $10^8 : 11^8$, i.e. nearly as 10 : 21.
 6. $8\frac{8}{9}$ ins. 8. Between 37 and 38. 9. 8. 10. 20.
 11. 22.

LVIII. (Page 278)

1. 34 feet. 2. 22 ft. 8 ins.

MISCELLANEOUS EXAMPLES

(Pages 279—293)

1. 15 lbs. wt. at an angle $\tan^{-1} \frac{4}{3}$ ($= 53^\circ 8'$) with the second force.
2. Each component is $57.735\dots$ lbs. wt.
3. 14.24 lbs. wt. 4. 50 and 86.6025... lbs. wt. 5. 3 feet.
6. 2 ozs. 8. $4\frac{1}{3}$ inches from A . 9. $7\frac{1}{3}$ inches.
11. It divides the line joining the centre to the middle point of the opposite side in the ratio 2 : 13.
12. 8 and 12 lbs. wt. 13. $9\frac{4}{5}$ lbs. wt. 14. $W = P$.
15. The required point divides the distance between the two extreme strings in the ratio 13 : 49.
16. $\frac{1}{3}\sqrt{3}$ lbs. wt. 18. 18 inches; 4 inches. 19. $26\frac{2}{3}$.
20. $\frac{2}{11}$. 21. $10\frac{935}{1344}$. 22. 40.
23. $\cos^{-1}(-\frac{1}{4})$, i.e. $104^\circ 29'$. 24. $2\sqrt{3}$ and $\sqrt{3}$ lbs. wt.
27. $8\frac{1}{2}$ lbs. wt.
28. The numbers marked on the machine must each be increased by $\frac{xW}{y10}$, where x and y are respectively the distances of the centre of gravity of the machine and its end from the fulcrum, and W is the weight of the machine.
29. 7766. 31. .446. 32. .11, .34, and .47 approx.
33. $a = 4.125$; $b = .01125$. 34. $a = 5.3$; $b = .097$ approx.
35. 120 lbs.; 2 ft. 37. $2 \sin^{-1} \frac{1}{3} = 38^\circ 56'$. 38. $\sqrt{6} : 1$.
40. 2 ft. $9\frac{1}{2}$ ins. from the lower end of the butt.
43. 209.26 tons wt. 44. 250π ft.-lbs. per sec. $= 1.43$ H.P.
45. $1\frac{3}{2}$ cwt. 46. 37.6 lbs. wt.; 50, $62\frac{1}{2}$, $68\frac{2}{11}$ and $74\frac{2}{7}$.
47. $160\pi = 502\frac{6}{7}$; $186\frac{2}{3}$; $\frac{7}{6\pi} = .371$. 48. 144 feet.
49. 15 secs. 50. 96 ft. per sec.; zero.
51. The first will have fallen through one-quarter of the height of the tower.
52. 2 secs. 53. $7\frac{1}{9}$ ozs. wt.; $5\frac{5}{9}$ ozs. wt. 55. $6\frac{7}{8}$ tons.
56. 896,000 units of impulse; 4 ft. 58. $\frac{3h}{8}$.
59. $\sqrt{gh}$, $\sqrt{gh}$, and 0, where h is the height of the plane.

61. $\frac{s}{30n}$. 62. South-west. 64. '938 ton wt. 65. $1\frac{1}{8}$ inches
67. At the end of time $\frac{1}{g}(u + \frac{1}{2}gt)$ from the starting of the first particle, and at a height of $\frac{1}{2g}(u^2 - \frac{1}{4}g^2t^2)$.
71. $\frac{5}{7}$ ft.; $\frac{1}{14}\sqrt{10}$ ($=.226\dots$) sec.; $\frac{3}{4}$.
73. $80\sqrt{110}$ ($=839.04$) ft. per sec.; $48\sqrt{110}$ ($=503.4$) ft. per sec.
74. (i) 45° ; (ii) 30° . 75. $65\frac{5}{8}$ ft.; $5\frac{5}{8}$ secs.
77. $1\frac{3}{16}$ and $2\frac{7}{16}$ ft. per sec. per sec.; $33\frac{1}{2}$ ft. per sec.
79. 14.4 secs. 81. $135\frac{3}{4}$ lbs. wt.
82. $\frac{g}{6}$ downwards; $\frac{g}{15}$ upwards.
83. 10 lbs. wt. eastward; 250 ft.-lb. units. 84. 15° or 75° .
86. 24×10^6 and $\frac{3}{14} \times 10^6$ ft.-lb.-sec. units; $\frac{3}{4} \times 10^6$ and $\frac{3}{448} \times 10^6$ lbs. wt. respectively.
88. $\frac{17}{60}$ sec.; $25\frac{5}{16}$ ft.-lb.-sec. units.
90. $\frac{7}{2}(\sqrt{2}-1)$, i.e. 1.45, secs. before the first stone starts.
91. The first by 30 secs.; the second by 30 secs.
92. About 22 feet behind the object.
93. $53^\circ 8'$ East of North; 4 mins.
94. $2\sqrt{34}$ ($=11.662$) miles per hour at $30^\circ 58'$ North of East; $\frac{15}{34}\sqrt{34}$ ($=2.5725$) miles.
95. 10.62 miles per hour at $3^\circ 16'$ South of West.
96. Approx. 15 miles per hour from the South.
97. $12\sqrt{2}$ ($=16.97$) knots.
98. Upstream at $72^\circ 33'$ with the bank; $\frac{15}{91}\sqrt{91}$ ($=1.57$) mins.
99. $7\frac{1}{2}$ mins.; at $53^\circ 8'$ with the direction of the stream;
- $\frac{30}{7}\sqrt{7}$ ($=11.34$) mins.; up stream at $\cos^{-1}\frac{3}{4}$ ($=41^\circ 25'$) with the bank.
100. 35 miles per hour at $\tan^{-1}\frac{4}{3}$ ($=53^\circ 8'$) with the direction of the first car.
101. $\frac{3125}{576}\pi$ ($=17.05$) lbs. wt. 102. $89\frac{31}{41}$ yds.
103. .6 gramme per cub. cm.; $1666\frac{2}{3}$ cub. cms. 104. $\frac{50}{63}$.
105. 357π ($=1122$) grammes. 106. 11:9. 109. 29.4 cms.
110. 32000 cub. ft. 111. $\frac{375\pi}{64} - 8$ = about 10.4 ozs. 112. 1.32.
113. .75. 114. 1.5. 115. $\frac{8}{305}$ of the original amount.
116. 5.05... inches. 117. 337:342. 118. 40 ton

20. $11\frac{3}{7}$ ins.; $\frac{25\pi}{72}$ ($= 1.09$) ozs. 121. $\frac{8}{13}$.
 22. $.64\dots$; $27\frac{9}{533}$ cub. ins. 123. $5\frac{55}{329}$ inches. 124. $19\frac{55}{83}$ cms.
 126. 16. 129. It rises through a distance $\frac{(\sigma_1 - \rho)(\sigma_3 - \sigma_2)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} h$.
 132. $1.0947\dots$ 133. $\frac{15}{32}$ inch. 134. 150 lbs. wt.
 136. 1020 cms.; 2,665,600. 138. 57.92 grammes.
 139. 15 grammes. 140. $\frac{44}{7}$. 142. 6248:1377.
 143. 2.684 metres. 144. 3.56 cms. 145. $17\frac{22}{31}$ inches.
 147. $72\frac{11}{37}$ cms. 148. 10 litres. 149. 15 atmospheres.
 150. $766\frac{2}{3}$ 151. 76 cms. 152. 6.514 cms.
 153. (1) $\frac{1}{2}r\sqrt{2}$; (2) r , where r is the radius of the common base.

**SEVEN
DAY
BOOK**